Some Characterizations of Generalized Top Trading Cycles*

Açelya Altuntaş† William Phan‡ Yuki Tamura§

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Abstract

Consider object exchange problems when each agent may be endowed with and consume more than one object. For most domains of preferences, no rule satisfies *efficiency*, the *endowment lower bound*, and *strategy-proofness*. Insisting on the first two properties, we explore the extent to which weaker incentive compatibility can be achieved. Motivated by behavioral and computational considerations as well as online mechanisms, we define several forms of manipulation. We consider the lexicographic domain of preferences, and provide four characterizations of Generalized Top Trading Cycles based on properties concerning immunity from heuristic and identity-splitting manipulations. We also show that this establishes a boundary with respect to incentive compatibility—minimal strengthening results in impossibility.

JEL Classification: C70, C78, D47, D70.

Keywords: endowment manipulation, heuristic manipulation, strategy-proofness, object exchange, Generalized Top Trading Cycles.

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†Deakin University. Contact: a.altuntas@deakin.edu.au.

‡North Carolina State University. Contact: wphan@ncsu.edu.

§New York University Abu Dhabi. Contact: yuki.tamura@nyu.edu.

1 Introduction

Consider a group of agents each of whom is endowed with indivisible goods, called "objects". Agents consume and have preferences over bundles of objects. Beneficial exchanges may be possible, and so the issue arises of redistributing the objects so as to achieve *(Pareto-) efficiency* and possibly other desirable properties. We focus on "lexicographic" preferences wherein an agent's preference relation over individual objects determines their preferences over the bundles.[1] Thus, a social planner needs only to elicit preferences over the objects.

We give several examples of this type of problem: Each employee at a company may have an initial set of tasks or shifts, and they may prefer to exchange them. Employees may have expertise or responsibilities for certain tasks, and so strongly (thus, lexicographically) prefer some to others. Students at a university wish to reschedule their courses by exchanging their assigned slots (Bichler et al., 2021; Budish, 2011; Budish and Cantillon, 2012). In the liver exchange program, a patient has donors who are possibly blood-type incompatible with them and thus seeks to exchange with others in the program (Ergin et al., 2020); beyond compatibility, they have preferences based on age, liver size, etc. of other potential donors.

We are interested in rules that satisfy the following three properties: *efficiency*, a participation requirement that each agent find their assignment at least as desirable as their endowment, the *endowment lower bound*,[2] and a strategic requirement that no agent ever benefit by misrepresenting their preference relation, *strategy-proofness*. Unfortunately, on the domain of lexicographic preferences, no rule satisfies these three properties (Remark 1).

Agents may not consider *all* possible misrepresentations of their preferences. In practice, an agent often uses heuristics to make a decision (Tversky and Kahneman, 1974;

[1]Lexicographic preferences have the feature that a preference relation over the objects pins down a unique preferences over the bundles. This is a practical advantage from an implementation perspective: as the number of objects increases, asking agents to rank the numerous bundles is increasingly prohibitive. Even though other domains (e.g. responsive) are more flexible, they would require agents to submit more information on their preferences.

[2]This property is also known as "individual rationality".

Kahneman et al., 1982; Gilovich et al., 2002). Instead of analyzing intricacies of a rule and others' preferences, an agent may only consider simple manipulations. Mennle et al. (2015) provide empirical evidence that agents indeed rely on "close by" lies to search for manipulations. Aside from simplicity, other reasons for local manipulation possibly include salience, credibility (too large of a lie may be detectable), and guilt.

We consider three forms of heuristic misrepresentations of preferences. First, starting from an agent's true preference relation over individual objects, they may *drop one object to the bottom* in their preference relation. A rule is *drop strategy-proof* if no agent benefits from doing so. In terms of the number of possible manipulations of preferences, for m objects, there are $m-1$ drop misrepresentations while there are $m!-1$ possible misrepresentations if we consider *strategy-proofness*. Thus especially in large economies, *drop strategy-proofness* is a much weaker property than *strategy-proofness*. Nevertheless, it turns out that a rule satisfies *efficiency*, the *strong endowment lower bound*, and *drop strategy-proofness* if and only if it is "Generalized Top Trading Cycles" (GTTC) (Theorem 1).

GTTC is an intuitive myopic extension of Gale's Top Trading Cycles (Shapley and Scarf, 1974) to the case where each agent may be endowed with and consume more than one object. Similar to TTC, at each step, each remaining agent points to their most preferred object among the available ones. However, a major departure from TTC is that because an agent may own more than one object, if they have not traded away all the objects in their endowment, they remain in the process. Hence, each agent receives as many objects as they own.[3]

We may define a family of incentive properties considering subsequently more sophisticated thinking. Instead of just once, an agent may perform a k-length sequence of drop manipulations. A rule is *k-drop strategy-proof* if it is immune to such a manipulation. A larger value of k results in a stronger property, and when k is the number of objects less one, we reach (full) *strategy-proofness*. It turns out that GTTC is at a boundary of what can be achieved in terms of non-manipulability of an agent's preference relation. Even only

[3]GTTC is also known as Augmented Top Trading Cycles.

2-drop strategy-proofness is incompatible with *efficiency* and the *endowment lower bound* (Proposition 2).

Second, an agent may *swap two adjacent objects* in their preference relation. A rule is *adjacent strategy-proof* if no agent benefits by such a swap (Carroll, 2012; Sato, 2013). Unfortunately, no rule satisfies *efficiency*, the *endowment lower bound*, and *adjacent strategy-proofness* (Proposition 2). However, we recover a positive result by weakening *adjacent strategy-proofness* as follows. Consider two adjacent objects a and b in agent i's preference relation. A rule satisfies *upper invariance* if agent i receives an object that is preferred to both a and b if and only if they receive the same object when they swap their preference between a and b (Hashimoto et al., 2014).[4][5][6] A rule satisfies *efficiency*, the *strong endowment lower bound*, and *upper invariance* if and only if it is GTTC (Theorem 2).

Third, an agent may *push one object up to the top* in their preference relation.[7] A rule is *push-up strategy-proof* if no agent benefits by pushing any object up to the top in their preference relation. Unfortunately, no *efficient* rule that satisfies the *endowment lower bound* is either (i) *drop strategy-proof* and *push-up strategy-proof*, (ii) *upper invariant* and *push-up strategy-proof*, or (iii) *2-push-up strategy-proof* (Proposition 2).

Next, an agent may manipulate a rule not only by misrepresenting their preference

[4]This property is also known as "weak invariance".

[5]In the context of probabilistic assignment, *strategy-proofness* can be decomposed into *upper invariance* and other two properties, "lower invariance" and "swap monotonicity" (Mennle and Seuken, 2021).

[6]*Upper invariance* is also related in spirit to dynamic mechanisms that 1) partially finalize the allocation intermittently, and 2) elicit preference information only over *remaining* objects. Bó and Hakimov (2020), Dur et al. (2018), and Klijn et al. (2019) in fact find benefits to dynamic variants of the Deferred Acceptance rule in school choice. Li (2017) shows that the *obviously strategy-proof* dynamic variant of Random Serial Dictatorship in object allocation problems indeed induces more truth-telling. Hakimov and Raghavan (2020) formalize notions of verifiability and transparency of mechanisms, and provide experimental evidence that these affect participant behavior.

[7]Pápai (2000) defines a more general class of push-ups where an object a is not necessarily pushed-up to the top, and the order of objects above a can be re-arranged. They make repeated use of this construction in their characterization of hierarchical exchange rules. Jaramillo and Manjunath (2012) consider object re-allocation with indifferences and define a "local" version where an object is pushed just above the objects in their indifference class.

relation but also through their endowment. For example, in online environments, agents' identities and endowments are not necessarily public information. An agent may very well opt to join with a *particular identity* as well as with only a part of their endowment. Furthermore, nothing precludes the agent from appearing multiple times with different identities each associated with a subset of their endowment. *Splitting invariance* requires that no agent affect their assignment by splitting their endowment. We show that a rule satisfies *efficiency*, the *endowment lower bound*, *upper invariance*, and *splitting invariance* if and only if it is GTTC (Theorem 3).

Finally, an agent may misrepresent their preference relation by dropping a *subset* of objects. These types of manipulations have been studied by Biró et al. (2021) for a related environment. We define two natural forms of such manipulations. In one version, an agent drops a subset of objects that they do not own to the bottom. A rule is *subset total drop strategy-proof* if it is immune to such a strategic move. A parallel characterization to Theorem 1 holds: A rule satisfies *efficiency*, the *strong endowment lower bound*, and *subset total drop strategy-proofness* if and only if it is GTTC (Theorem 4). The second version considers a more general class of drops. When an agent drops an object a (that they do not own), it must end up below at least *one* new endowment object. This must be true for each object in the subset that they drop. A rule is *subset drop strategy-proof* if it is immune to such a manipulation. Unfortunately, no *efficient* rule meeting the *endowment lower bound* is *strong subset drop strategy-proof* (Proposition 4).

While the above analysis has focused on axioms, we also show that the lexicographic domain plays an important role in terms of the scope of GTTC. Consider a domain that contains that of lexicographic preferences and further satisfies preference monotonicity with respect to superset bundles. GTTC is still well-defined, but now violates *efficiency* (Proposition 3). So lexicographic preferences is a maximal one on the domain of monotonic preferences for GTTC to be *efficient*. Similarly, responsive preferences is a maximal domain for GTTC to satisfy the *endowment lower bound* (Proposition 3).

1.1 Related Literature

For object exchange problems when each agent may own and consume more than one object, and the preference domain satisfies a richness condition, no rule satisfies *efficiency*, the *endowment lower bound*, and *strategy-proofness* (Sönmez, 1999). This impossibility result holds for other environments. For instance, suppose that there are several "types" of objects, and each agent owns and consumes one object of each type. Preferences are defined over bundles containing one object of each type. If preferences are additively separable, no rule satisfies the above three properties (Konishi et al., 2001).[8] Another important class of problems is when each agent owns one type of object but may have more than one copy. If each agent has responsive preferences and receives as many objects as they own, again no rule satisfies the three properties (Biró et al., 2021).

On the other hand, for some situations, rules that satisfy *efficiency*, the *endowment lower bound*, and *strategy-proofness* exist. In the model with several types of objects, on the domain where each preference relation is a lexicographic extension of a "CP-net", there is such a rule (Sikdar et al., 2017). In the model of Biró et al. (2021), when agents have lexicographic preferences, the three properties characterize a rule similar to GTTC.[9] When agents have "trichotomous" or "dichotomous" preferences, there is possibility (Manjunath and Westkamp, 2021; Andersson et al., 2021). Suppose that objects are separated into different markets, and each object can be traded with another object only if they are in

[8]For this class of problems, the core is in general empty (Moulin, 1995; Konishi et al., 2001). Given this negative result, alternative solution to the core, the "coordinate-wise competitive equilibrium", and the rule that selects the coordinate-wise competitive allocation, the "coordinate-wise core", were introduced (Konishi et al., 2001; Wako, 2005). The coordinate-wise core is the only rule satisfying a weak form of *efficiency*, the *endowment lower bound*, *strategy-proofness*, and an invariance property, "non-bossiness" (Miyagawa, 1997). Also, no *strategy-proof* rule Pareto-dominates this rule (Klaus, 2008; Anno and Kurino, 2016). On the other hand, if objects are separated into different and ordered markets, and agents have lexicographic preferences, the coordinate-wise core satisfies *efficiency*, the *endowment lower bound*, and *strategy-proofness*.

[9]Their Circulation Top Trading Cycles takes as input agents' preferences as well as how many copies of their own endowment each agent has. Multiple copies of objects may be traded along each cycle.

the same market; then again the properties are compatible (Pápai, 2003, 2007).[10]

From a computational complexity standpoint, manipulation of GTTC on the lexicographic domain is difficult. Fujita et al. (2015) show that GTTC selects from the core and is NP-hard to manipulate. Considering parameterized complexity, it is W[P]-hard to manipulate (as well as in groups) (Phan and Purcell, 2019).

Our work also relates to the study of manipulation through endowments. These types of properties turn out to be very demanding. Even on restricted preference domains, no rule that satisfies *efficiency* and the *endowment lower bound* is immune to endowment manipulation in general (Atlamaz and Klaus, 2007; Atlamaz and Thomson, 2006; Bu et al., 2014).[11] In particular, on the domain of "size monotonic" preferences, no rule that satisfies *efficiency* and the *endowment lower bound* is *splitting-proof* (Bu et al., 2014).[12] On the other hand, when we drop *efficiency*, a rule exists that satisfies not only the *endowment lower bound* and *splitting-proofness*, but also *strategy-proofness* (Todo et al., 2014).

The paper is organized as follows. In Section 2, we define the model and properties of rules. In Section 3, we define GTTC. Then we state characterizations of this rule as well as some impossibility results. In Section 4, we discuss manipulation of an agent by dropping multiple objects. In Section 5, we conclude. All proofs are collected in Appendices.

[10]Due to feasibility constraints, notions of *efficiency* in Pápai (2003) and Pápai (2007) are weaker than the usual definition.

[11]Endowment manipulation has played a key role in other classes of problems, and we briefly mention several. Thomson (2010a) and Thomson (2010b) consider the problems of allocating infinitely divisible goods; Moulin (2007) and Moulin (2008), scheduling; Conitzer (2008) and Todo et al. (2011), voting; Yokoo et al. (2004a), combinatorial online auctions; Leshno and Strack (2020), cryptocurrencies such as Bitcoin; Aumann and Shapley (2015) and Sprumont (2005), characterizing the Aumann-Shapley rule in cost-sharing; and O'Neill (1982) and Ju et al. (2007), characterizing the proportional rule in the bankruptcy problem.

[12]A similar concept of manipulation, called "false-name-proofness", has been also studied (Yokoo et al., 2004b; Todo and Conitzer, 2013).

2 Model

Let $\mathbb{N}$ be the set of potential agents, and $\mathbb{O}$ be the set of potential objects. Each agent i in $\mathbb{N}$ is endowed with a finite set ω_i of objects in $\mathbb{O}$, no two agents being endowed with identical objects. For each $o \in \mathbb{O}$, let $\omega(o) \in \mathbb{N}$ be the owner of object o.

Each agent i in $\mathbb{N}$ has a **lexicographic** preference relation R_i over non-empty subsets of $\mathbb{O}$, where we denote by P_i the strict component of R_i: Each agent has a strict preference relation over the individual objects and uses this relation to compare each pair of bundles. Let $X, Y \in \mathbb{O}$. If an agent i prefers the most preferred object in X to that in Y, then they prefer X to Y. If these objects are the same, they compare the second most preferred object in each bundle. If the second most preferred object in X is better than that in Y, then the agent prefers X to Y, and so on. Finally, if Y is a subset of X, the agent prefers X to Y. Formally, for each $i \in N$ and each pair $X, Y \subseteq \mathbb{O}$, $X \ P_i \ Y$ if and only if there is $o \in X \setminus Y$ such that (i) for each $o' \in Y$, if $o' \ P_i \ o$, then $o' \in X$, and (ii) for each $o' \in Y \setminus X$, $o \ P_i \ o'$.[13] Let $\mathcal{R}$ be the set of lexicographic preference relations.

Each agent's preference relation over individual objects induces a unique preference relation over bundles of objects. That is, no two preference relations over individual objects induces the same preference relation over bundles. Hence, it is sufficient for each agent to submit their preference relation over individual objects, and we assume that they do so. We use R_i to describe agent i's preferences over individual objects as well as the induced preferences over bundles. We write $R_i : o, o', o''$ to mean that o is preferred to o', o' is preferred to o''. For each $o \in \mathbb{O}$, the **upper contour set at $\boldsymbol{R_i}$ of $\boldsymbol{o}$** is $U(R_i, o) \equiv \{o' \in \mathbb{O} : o' \ R_i \ o\}$, and the **strict upper contour set at $\boldsymbol{R_i}$ of $\boldsymbol{o}$** is $SU(R_i, o) \equiv \{o' \in \mathbb{O} : o' \ P_i \ o\}$. The **strict lower contour set at $\boldsymbol{R_i}$ of $\boldsymbol{o}$** $SL(R_i, o)$ is defined analogously. For each $O \subseteq \mathbb{O}$, we denote by $T(R_i, O)$ the **most preferred object at $\boldsymbol{R_i}$ in $\boldsymbol{O}$**.

In each economy, a subset of potential agents are present, each being identified by their endowment and their preference relation. Hence, an **economy** is a list (N, R, ω), where $N \subseteq \mathbb{N}$, $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$, and $\omega \equiv (\omega_i)_{i \in N} \in \mathbb{O}^N$. Let $\mathcal{E}$ be the set of economies.

[13] For brevity, we write a singleton set $\{o\}$ simply as o, and a bundle $\{a, b, c\}$ also as abc.

For each $(N, R, \omega) \in \mathcal{E}$, if $x \equiv (x_i)_{i \in N}$ is a list of subsets of $\bigcup_{i \in N} \omega_i$ such that for each pair $i, j \in N$, $x_i \cap x_j = \emptyset$, and $\bigcup_{i \in N} x_i = \bigcup_{i \in N} \omega_i$, we say that x is an **allocation of (N, R, ω)**. Note that an endowment profile is an allocation. For each $(N, R, \omega) \in \mathcal{E}$, let $\mathcal{X}(N, R, \omega)$ be the set of allocations of (N, R, ω), and let $\mathcal{X} \equiv \bigcup_{(N,R,\omega) \in \mathcal{E}} \mathcal{X}(N, R, \omega)$. A **rule** is a single-valued mapping $\varphi : \mathcal{E} \to \mathcal{X}$ such that for each $(N, R, \omega) \in \mathcal{E}$, we have $\varphi(N, R, \omega) \in \mathcal{X}(N, R, \omega)$. For each $(N, R, \omega) \in \mathcal{E}$, a rule selects an allocation for N that is independent of preference relations over the objects that are not present, and of the preference relations of the agents who are not present. For each $(N, R, \omega) \in \mathcal{E}$ and each $i \in N$, agent i's assignment given by φ at (N, R, ω) is denoted by $\varphi_i(N, R, \omega)$.

We define basic properties of rules. First property states that for each economy, the selected allocation should be such that there is no other allocation that all agents find at least as desirable and at least one agent prefers:

Efficiency For each $(N, R, \omega) \in \mathcal{E}$, there is no $x' \in \mathcal{X}(N, R, \omega)$ such that

1. for each $i \in N$, $x'_i \; R_i \; \varphi_i(N, R, \omega)$, and

2. there is $j \in N$ such that $x'_j \; P_i \; \varphi_j(N, R, \omega)$.

If there is such an x', then we say that x' Pareto-dominates $\varphi(N, R, \omega)$ at (N, R, ω).

Our second property states that for each economy, the selected allocation should be such that each agent finds their assignment at least as desirable as their endowment:

Endowment Lower Bound For each $(N, R, \omega) \in \mathcal{E}$, and each $i \in N$,

$$\varphi_i(N, R, \omega) \; R_i \; \omega_i.$$

The next property strengthens the *endowment lower bound*: There is a one-to-one relation between the objects in an agent's assignment and objects in their endowment. In particular, the agent should find each object in their assignment at least as desirable as the corresponding object in their endowment:

Strong Endowment Lower Bound For each $(N, R, \omega) \in \mathcal{E}$ and each $i \in N$, there is a bijection $\sigma : \omega_i \to \varphi_i(N, R, \omega)$ such that for each $o \in \omega_i$,

$$\sigma(o) \; R_i \; o.$$

Hence, this property requires that each agent is assigned as many objects as they own.[14]

The next property states that no agent should ever benefit by misrepresenting their preference relation:

Strategy-proofness For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$,

$$\varphi_i(N, R, \omega) \; R_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

Remark 1. It is known that on the domain of lexicographic preferences, no rule satisfies *efficiency*, the *endowment lower bound*, and *strategy-proofness* (Todo et al., 2014).[15]

2.1 Heuristic Manipulations

Motivated by evidence that agents attempt to manipulate rules with natural heuristics, we define several families of such. Starting from an agent's true preference relation (over the individual objects), consider the following simple "one-step" operations: An agent selects *one* object and

- **drops** it to the bottom of their preferences,
- **pushes** it **up** to the top of their preferences, or
- **swaps** it with an **adjacent** object in their preferences.

Each operation is intuitive and results in a subsequent preference relation. Formally, let $(N, R, \omega) \in \mathcal{E}$ and $i \in N$. We say that $R'_i \in \mathcal{R}$ is a

- **drop for $\boldsymbol{R_i}$** if there is $o \in \mathbb{O}$ such that
    - for each $o' \in \mathbb{O}$, $o' \; R'_i \; o$, and

[14] A requirement that each agent is assigned as many objects as they own is called "balance". On the domain of lexicographic preferences, the *strong endowment lower bound* implies both the *endowment lower bound* and *balance*. On the other hand, the combination of the *endowment lower bound* and *balance* does not imply the *strong endowment lower bound*.

[15] Their definition of *strategy-proofness* allows an agent to withhold part of their endowment; however, their impossibility result does not require manipulation by withholding.

- for each pair $o', o'' \in \mathbb{O} \setminus \{o\}$, $o' \; R_i \; o'' \Leftrightarrow o' \; R'_i \; o''$,

- **push-up for $\boldsymbol{R_i}$** if there is $o \in \mathbb{O}$ such that
  - for each $o' \in \mathbb{O}$, $o \; R'_i \; o'$, and
  - for each pair $o', o'' \in \mathbb{O} \setminus \{o\}$, $o' \; R_i \; o'' \Leftrightarrow o' \; R'_i \; o''$,
- **adjacent swap for $\boldsymbol{R_i}$** if there is a pair $o, o' \in \mathbb{O}$ such that
  - $o \; P_i \; o'$,
  - there is no $o'' \in \mathbb{O}$ such that $o \; P_i \; o'' \; P_i \; o'$,
  - $o' \; P'_i \; o$,
  - there is no $o'' \in \mathbb{O}$ such that $o' \; P'_i \; o'' \; P'_i \; o$, and
  - for each $o'', o''' \in \mathbb{O} \setminus \{o, o'\}$, $o'' \; R_i \; o''' \Leftrightarrow o'' \; R'_i \; o'''$.

We define an incentive compatibility associated with each of these operations.

Drop Strategy-proofness For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$ where R'_i is a drop for R_i,

$$\varphi_i(N, R, \omega) \; R_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

We define **push-up strategy-proofness** and **adjacent strategy-proofness** analogously by replacing the requirement on R'_i.

If an agent comprehends a drop for their preference relation, one may think that they can also comprehend a second drop for their preference relation. We define the subsequent drops for any sequence of length k as follows. A preference relation R'_i is a **$\boldsymbol{k}$-drop for $\boldsymbol{R_i}$** if there is a sequence $R_i^1, \ldots, R_i^k$ where $R'_i \equiv R_i^k$ is a drop for R_i^{k-1}, R_i^{k-1} is a drop for R_i^{k-2}, ..., and R_i^2 is a drop for $R_i^1 \equiv R_i$.

$\boldsymbol{k}$-drop Strategy-proofness For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, each $R'_i \in \mathcal{R}$, and each $\bar{k} \leq k$ such that R'_i is a $\bar{k}$-drop for R_i,

$$\varphi_i(N, R, \omega) \; R_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

We define ***k*-push-up strategy-proofness** and ***k*-adjacent strategy-proofness** analogously by replacing the requirement on R'_i. Intuitively, larger k represents a stronger condition.

At first glance the logical relations between these families of properties are unclear. For example, how much do we have to strengthen *k-drop strategy-proofness* until it implies *adjacent strategy-proofness*? When $k = |\mathbb{O}| - 1$, clearly *k-drop strategy-proofness* is equivalent to (full) *strategy-proofness*. It turns out that if $k < |\mathbb{O}| - 1$, *k-drop strategy-proofness* does not imply even *adjacent strategy-proofness* or *push-up strategy-proofness*. Hence, only the maximum strength of *k-drop strategy-proofness* implies each one of *adjacent strategy-proofness* and *push-up strategy-proofness*. An analogous statement holds for *k-push-up strategy-proofness* implying each one of *adjacent strategy-proofness* and *drop strategy-proofness*. Finally, it turns out that *adjacent strategy-proofness* is equivalent to *strategy-proofness*.

Proposition 1. Some logical relations between the properties.

1. *k-drop strategy-proofness* implies each one of *push-up strategy-proofness*, *adjacent strategy-proofness*, and *strategy-proofness* if and only if $k \geq |\mathbb{O}| - 1$.

2. *k-push-up strategy-proofness* implies each one of *drop strategy-proofness*, *adjacent strategy-proofness*, and *strategy-proofness* if and only if $k \geq |\mathbb{O}| - 1$.

3. *Adjacent strategy-proofness* is equivalent to *strategy-proofness*.[16]

The proof of Proposition 1 is in Appendix A.

The third point in Proposition 1 implies that *adjacent strategy-proofness* is incompatible with *efficiency* and the *endowment lower bound*.[17] Following this result, we consider a property that is implied by *adjacent strategy-proofness* and that has been of interest in the literature. Suppose that an agent i reports an adjacent swap for their true preferences R_i, say they swap objects a and b, where $a\ R_i\ b$. We denote by R'_i the resulting preference

[16]This result cannot be deduced from Proposition 2 of Carroll (2012) – the union of all types (ordinal preferences in the lexicographic domain) does not form a convex set.

[17]We state this impossibility result in Proposition 2.

relation. The following property requires that for each object o in the strict upper contour set at R_i of a, agent i is assigned o when they report R'_i if and only if they receive o when they report R_i.[18]

Upper Invariance For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, each pair $a, b \in \bigcup_{j \in N} \omega_j$, and each $R'_i \in \mathcal{R}$, if R'_i is an adjacent swap for R_i, $a \; P_i \; b$, and $b \; P'_i \; a$, then for each $o \in SU(R_i, a)$,

$$o \in \varphi_i(N, R'_i, R_{-i}, \omega) \iff o \in \varphi_i(N, R, \omega).$$

Lemma 1. *Adjacent strategy-proofness* implies *upper invariance.*

The proof of Lemma 1 is in Appendix B.

So far we studied several forms of misrepresentation of an agent's preference relation. When an agent's identity is not known a priori, they must report their name, characteristics, and endowment. This feature is prominent in, for example, online mechanisms and leaves room for manipulations (via misreporting, say, one's endowment) that is separate from preference misrepresentation.

The following property requires that if an agent "splits" into several agents and partitions their endowment amongst the new identities, then the rule should assign this agent the same assignment.

Splitting Invariance Let $(N, R, \omega), (N', R', \omega') \in \mathcal{E}$ be such that

- for each $i \in N$, there is $N^i \subset N'$ such that for each $j \in N^i$, $R'_j = R_i$, and $\bigcup \omega'_j = \omega_i$,
- for each pair $i, j \in N$, $N^i \cap N^j = \emptyset$, and
- $\bigcup N^i = N'$.

Then for each $i \in N$,

$$\varphi_i(N, R, \omega) = \bigcup_{j \in N^i} \varphi_j(N', R', \omega').$$

[18]In probabilistic allocation, Mennle and Seuken (2021) show that *strategy-proofness* is equivalent to the combination of this property and two other weak forms of non-manipulability. Each of Bogomolnaia and Heo (2012) and Hashimoto et al. (2014) provide a characterization of the Serial Rule defined by Bogomolnaia and Moulin (2001) using this property or its variants.

Lemma 2. The *endowment lower bound* and *splitting invariance* together imply the *strong endowment lower bound*.

The proof of Lemma 2 is in Appendix B.

Remark 2. Suppose that an agent is allowed to report different preference relations than their original one at the identities they created. A property that is immune to such a manipulation implies *strategy-proofness*.

3 Characterizations and Impossibilities

3.1 Generalized Top Trading Cycles

We define the rule central to our study. For each $(N, R, \omega) \in \mathcal{E}$, we use the following algorithm to determine the allocation recommended by **Generalized Top Trading Cycles (GTTC)**, G:

Step 1 Construct a directed graph as follows: The set of vertices is N and O. For each agent in the graph, there is a directed edge to their most preferred object in the graph according to their preference relation. For each object in the graph, there is a directed edge to its owner. At least one cycle exists. For each cycle, each agent i in the cycle receives the object a for which there is an edge from i to a.

Step $s > 1$ Construct a directed graph as follows: The set of vertices is the set of 1) agents who have not received as many objects as they own, and 2) objects that are not assigned to any agent. For each agent in the graph, there is a directed edge to their most preferred object in the graph according to their preference relation. For each object in the graph, there is a directed edge to its owner. At least one cycle exists. For each cycle, each agent i in the cycle receives the object a for which there is an edge from i to a.

Because there are finite numbers of agents and objects, the algorithm ends in finitely many steps.

Example 1. *Illustrating GTTC.* Let $N = \{1,2,3\}$ and $O = \{a,b,c,d,e,f\}$. Let $\omega = (ab, cde, f)$ and $R \in \mathcal{R}^N$ be such that

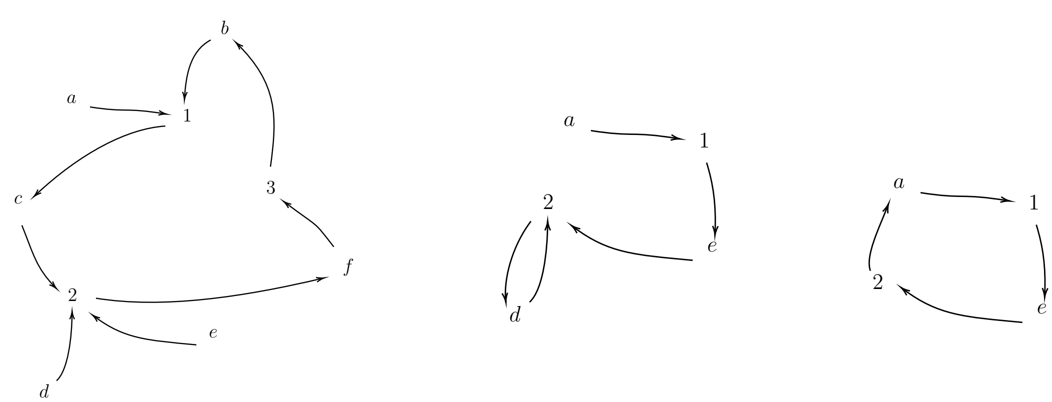
$$R_1 : \; c, \; e, \ldots$$
$$R_2 : \; f, \; d, \; a, \ldots$$
$$R_3 : \; b, \ldots.$$

Step 1 [Figure 1a] Each agent points to their most preferred object and each object points to its owner. Agents 1,2,3 and objects b, c, f form a cycle. Hence, each agent in the cycle receives the object that they point to. Since agent 3 receives as many objects as they own, they leave with their assignment.

Step 2 [Figure 1b] Each remaining agent points to their most preferred object among the available ones and each remaining object points to its owner. Agent 2 and object d form a cycle. Hence, agent 2 receives object d.

Step 3 [Figure 1c] Each remaining agent points to their most preferred object among the available ones and each remaining object points to its owner. Agent 1,2 and objects a, e form a cycle. Hence, each agent in the cycle receives the object that they have point to. Since each of agent 1 and agent 2 receives as many objects as they own, they leave with their assignments.

Since there is no remaining agent, the algorithm terminates and the allocation selected by GTTC at (N, R, ω) is $G(N, R, \omega) = (ce, adf, b)$.



(a) Step 1 (b) Step 2 (c) Step 3

Figure 1: Illustration of GTTC

Unfortunately, GTTC is not *strategy-proof* (Example 2).

Example 2. *GTTC is not strategy-proof.* Let $(N, R, \omega) \in \mathcal{E}$ be such that $N = \{1, 2\}$, $\omega = (a, bc)$, and

$$R_1 : b,\ c,\ a$$

$$R_2 : a,\ b,\ c.$$

Then,

$$G(N, R, \omega) = (b, ac).$$

Let $R'_2 \in \mathcal{R}$ be such that

$$R'_2 : b,\ a,\ c.$$

Then,

$$G(N, R_1, R'_2, \omega) = (c, ab).$$

Because

$$G_2(N, R_1, R'_2, \omega) = ab\ P_2\ ac = G_2(N, R, \omega),$$

GTTC is not *strategy-proof.*

In fact, Example 2 shows that GTTC is not even *2-drop strategy-proof.*

3.2 Characterizations of GTTC

Theorem 1. A rule satisfies *efficiency*, the *strong endowment lower bound*, and *drop strategy-proofness* if and only if it is GTTC.

Consider an economy (N, R, ω). The proof proceeds by induction on a sequence of classes of preference profiles leading up to R. The first class is comprised of preference profiles derived from R as follows: Each agent in the *first cycle* in the GTTC algorithm for R ranks only the object they point to in the cycle above their own endowment—all other objects are dropped. The second class of preference profiles "adds back in" exactly *one* object for one agent that was dropped. Along this sequence, we show that a rule satisfying the properties coincides with GTTC. Within each induction step, there is a sub-induction step to handle multiple cycles.

Theorem 2. A rule satisfies *efficiency*, the *strong endowment lower bound*, and *upper invariance* if and only if it is GTTC.

The proof follows closely to that of Theorem 1, and explains how to invoke *upper invariance* instead of *drop strategy-proofness* at the relevant steps.

Theorem 3. A rule satisfies *efficiency*, the *endowment lower bound*, *upper invariance*, and *splitting invariance* if and only if it is GTTC.

Concerning a proof of Theorem 3, by Lemma 2, it suffices to show that GTTC is *splitting invariant*. Nevertheless, we provide a direct proof of Theorem 3 in Appendix C. The idea of the proof is as follows. We first focus on economies where each agent owns one object. In such economies, GTTC is equivalent to TTC. Following the proof in Ma (1994), we show that if a rule satisfies *efficiency*, the *endowment lower bound*, and *upper invariance*, it is GTTC. Then by invoking *splitting invariance*, we show that for economies where an agent may own more than one object, if a rule satisfies the properties in Theorem 3, it is GTTC.

All proofs of statements in this section are in Appendix C, and we show independence of axioms in the characterizations in Appendix D. For Theorems 1 and 2, if we weaken the *strong endowment lower bound* to the *endowment lower bound*, another rule exists that differs from GTTC and satisfies all the properties in the theorems (Example 6 in Appendix D).

3.3 Impossibility Results

In some sense, GTTC represents the extent to which we can get incentive compatibility properties while insisting on *efficiency* and the *endowment lower bound*. For each of the theorems above, considering an addition or strengthening results in impossibility.

Proposition 2. No rule satisfies *efficiency*, the *endowment lower bound*, and

1. *upper invariance* and *push-up strategy-proofness*,
2. *drop strategy-proofness* and *push-up strategy-proofness*,
3. *2-drop strategy-proofness*,
4. *2-push-up strategy-proofness*, or
5. *adjacent strategy-proofness*.[19]

Proof of the proposition is provided in Appendix E. Each of these previous statements also holds for the subdomain of *single-peaked lexicographic preferences*: Each agent's preference relation over the individual objects satisfies the following condition. There is an order on the object set such that for each agent, there is an object such that the further an object is from the distinguished one, in either direction, the less desirable it is for the agent.

We also remark on the difficulty of applying GTTC on preference domains larger than the domain of lexicographic preferences. For example, on the domain of "responsive" preferences,[20] GTTC is no longer *efficient*.

Example 3. *On the domain of responsive preferences, GTTC is not efficient.* Let $(N, R, \omega) \in \mathcal{E}$ be such that $N = \{1, 2\}$, $\omega = (bc, ad)$, and

$$R_1 : \ldots, ad,\ bc, \ldots, a,\ b,\ c,\ d$$
$$R_2 : \ldots, bc,\ ad, \ldots, a,\ b,\ c,\ d.$$

Then

$$G(N, R, \omega) = (bc, ad).$$

Let $x \in \mathcal{X}(N, R, \omega)$ be such that

$$x = (ad, bc).$$

[19]This is a corollary of Statement 3 of Proposition 1 and the fact that *efficiency*, the *endowment lower bound*, and *strategy-proofness* are incompatible (Todo et al., 2014). We include this result and show that the same proof can be used for all the statements in Proposition 2.

[20]A preference relation over the sets of objects is **responsive** if for each pair $o, o' \in \mathbb{O}$ and each $O \subset \mathbb{O} \setminus \{o, o'\}$, $o\ P_i\ o' \iff O \cup \{o\}\ P_i\ O \cup \{o'\}$.

Because x Pareto-dominates $G(N, R, \omega)$ at R, G is not *efficient*.

We may generalize this to *any* domain that admits more flexibility than lexicographic preferences regarding consistency of preferences over singletons versus bundles. The myopic nature of GTTC clashes with this flexibility, leading to the loss of *efficiency*. We also include a similar statement for responsive preferences and the *endowment lower bound*.

Proposition 3. GTTC and maximal domains for two properties.

1. On the domain of monotonic preferences, the set of lexicographic preferences is a maximal domain on which GTTC is *efficient*.[21]

2. The set of responsive preferences is a maximal domain on which GTTC meets the *endowment lower bound*.

In other words, on the domain of monotonic preferences, if a preference domain strictly contains the domain of lexicographic preferences, then there is an economy such that each other agent has lexicographic preferences and the allocation selected by GTTC is not *efficient* at that economy. Similarly, if a preference domain strictly contains the domain of responsive preferences, then there is an economy such that the allocation selected by GTTC does not meet the *endowment lower bound* at that economy. The proof of the proposition is provided in Appendix D.

4 Manipulation By Dropping Multiple Objects

In this section, we consider manipulations whereby an agent drops a *subset* of objects below their endowment.[22] We define two natural versions; the latter defends against a

[21] A preference relation R_i is monotonic if for each pair $A, B \subseteq \mathbb{O}$, if $B \subset A$, then $A \; P_i \; B$.

[22] These properties follow in spirit to that of *dropping strategy-proofness* in Biró et al. (2021), although theirs does not directly apply here due to differences in the environments. In their model, each agent has one *type* of object with possible multiple copies, have preferences over types, and then responsive or lexicographic preferences over bundles consisting of various quantities of each type. A drop manipulation is defined as taking any subset of objects preferred to one's endowment and dropping it below the endowment. Our *subset drop strategy-proofness* is closest in spirit to their version.

superset of manipulations and is thus stronger. The first characterizes GTTC along with *efficiency* and the *strong endowment lower bound*. This result is parallel to Theorem 1, substituting out *drop strategy-proofness* for the new property. The second property is not compatible with the combination of *efficiency* and the *endowment lower bound*.

Consider an agent i's preference relation R_i. Suppose that they misrepresent their preference relation as follows. Agent i selects a subset of objects that they do not own, and drops them below *each* object in agent i's endowment (in any order). We can then define the associated incentive compatibility that is immune to such forms of manipulations. Consider a more general class of drop manipulations: When an agent drops an object, they drop it below at least *one* object in their endowment.

Formally, a preference relation R'_i is a **subset total drop for $\boldsymbol{R_i}$** if there is $X \subseteq \mathbb{O} \setminus \omega_i$ such that

- for each $o \in X$ and each $o' \notin X$, $o' \; P'_i \; o$, and
- for each pair $o, o' \in \mathbb{O} \setminus X$, $o \; R_i \; o' \Leftrightarrow o \; R'_i \; o'$.

The first condition states that each object in X are dropped to the bottom. The second condition states that the relative ranking of the objects that are not dropped remains the same.

A preference relation R'_i is a **subset drop for $\boldsymbol{R_i}$** if there is $X \subseteq \mathbb{O} \setminus \omega_i$ such that

- for each $o \in X$,
$$\{o' \in \omega_i : \; o' \; P'_i \; o\} \supsetneq \{o' \in \omega_i : \; o' \; P_i \; o\}$$
- for each pair $o, o' \in \mathbb{O} \setminus X$, $o \; R_i \; o' \Leftrightarrow o \; R'_i \; o'$.

The first condition states that each object in X is dropped below at least one additional object in their endowment. The second condition states that the relative ranking of the objects that are not dropped remains the same.

Subset Total Drop Strategy-proofness For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$ where R'_i is a subset total drop for R_i,

$$\varphi_i(N, R, \omega) \; R_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

Subset Drop Strategy-proofness For each $(N, R, \omega) \in \mathcal{E}$, each $i \in N$, and each $R'_i \in \mathcal{R}$ where R'_i is a subset drop for R_i,

$$\varphi_i(N, R, \omega) \; R_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

In *drop strategy-proofness*, an agent may drop an object that they own to the bottom of their preference relation. Such a misrepresention is not considered in a subset (total) drop; hence, neither of these two conditions imply *drop strategy-proofness*. On the other hand, *k-drop strategy-proofness* implies both if and only if $k = |\mathbb{O}| - 1$.

Theorem 4. A rule satisfies *efficiency*, the *strong endowment lower bound*, and *subset total drop strategy-proofness* if and only if it is GTTC.

An analogous argument to that in Lemma 3 shows that GTTC is *subset total drop strategy-proof*. The proof in the other direction is identical to that of Theorem 1. Observe that each time *drop strategy-proofness* is invoked, instead, *subset total drop strategy-proofness* can be used.[23] The same rules showing independence of the properties in Theorem 1 can used to show independence here (Appendix D). Finally, relaxing *strong endowment lower bound* to just the *endowment lower bound* results in additional rules aside from GTTC (Example 6).

Proposition 4. No rule satisfies *efficiency*, the *endowment lower bound*, and *subset drop strategy-proofness*.

The proof of Proposition 2 is also applied here. To see this, observe that in the proof, R'_1 is a subset drop for R_1, and R'_2 is a subset drop for R_2.

5 Conclusion

We studied object exchange problems where each agent may be endowed with and consume more than one object. We provide characterizations of GTTC based on several weak

[23]In fact, a weakening of *drop strategy-proofness* (that implies *subset total drop strategy-proofness*) can be used where we only allow an agent to misrepresent via a 1-drop of any object that is not in their endowment.

forms of incentive compatibility. Once an agent manipulates their preferences in more complex ways, no rule that satisfies *efficiency* and the *endowment lower bound* prevents such manipulation.

We conclude this paper with some open questions. First, even though we provide several logical relations of the properties concerning preference manipulation, we have not derived complete logical relations between them e.g. what combination of k*-drop* and k'*-push-up strategy-proofness* imply the other properties. Second, it is still unknown whether or not there is a rule that satisfies *efficiency*, the *endowment lower bound*, and *push-up strategy-proofness*. Third, we focus on the domain of lexicographic preferences. It would be interesting to see if we can enlarge the domain in such a way that GTTC is well-defined and satisfies desirable properties—we do not know a general condition of preference domains on which a rule exists that satisfies *efficiency* and the *endowment lower bound*, and is immune to some form of preference manipulation. Fourth, truth-telling of preferences is not a Nash equilibrium. Nash equilibria of the preference revelation game induced by GTTC and their associated allocations is unknown. A similar question holds for the endowment manipulation game induced by GTTC. Finally, GTTC is a selection from the core that satisfies *drop strategy-proofness*. We did not further explore questions regarding selections from the core that satisfy our various non-manipulability properties.

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A Proof of Proposition 1

Statement 1: *k-drop strategy-proofness* implies each one of *push-up strategy-proofness*, *adjacent strategy-proofness*, and *strategy-proofness* if and only if $k \geq |\mathbb{O}| - 1$.

This statement follows from Claim 1 and Example 4.

Claim 1. Let $i \in \mathbb{N}$ and $R_i \in \mathcal{R}$. Each $R'_i \in \mathcal{R}$ is a $|\mathbb{O}| - 1$-drop for R_i.

Proof of Claim 1. Let $i \in \mathbb{N}$ and $R_i \in \mathcal{R}$. We label the objects in $\mathbb{O}$ in such a way that

$$R_i : a_1, a_2, \ldots, a_{|\mathbb{O}|}.$$

Let $R'_i : a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(|\mathbb{O}|)} \in \mathcal{R}$ for some bijection $\sigma : \{1, \ldots, |\mathbb{O}|\} \to \{1, \ldots, |\mathbb{O}|\}$. Consider the following sequence of $|\mathbb{O}| - 1$ drops starting from R_i: First drop $a_{\sigma(2)}$, then drop $a_{\sigma(3)}$, …, finally drop $a_{\sigma(|\mathbb{O}|)}$ to reach R'_i. Since the latter was arbitrarily chosen, $|\mathbb{O}| - 1$ drops are needed to cover all possible preference relations. $\square$

Consequently, when $k = |\mathbb{O}| - 1$, *k-drop strategy-proofness* implies *strategy-proofness*, and thereby both *push-up* and *adjacent strategy-proofness*.

We now show that at least $k = |\mathbb{O}| - 1$ is required to have the implication in Statement 1. Suppose that $k = |\mathbb{O}| - 2$. The following rule is *k-drop strategy-proof* but not *push-up strategy-proof* or *adjacent strategy-proof*.

Example 4. *A rule that is k-drop strategy-proof where $k = |\mathbb{O}| - 2$, but not push-up strategy-proof or adjacent strategy-proof.* Let $(N, R, \omega) \in \mathcal{E}$ and $i, j \in \mathbb{N}$ be such that $N = \{i, j\}$ and $\omega_i \cup \omega_j = \mathbb{O}$. Let $a, b \in \mathbb{O}$. Let φ be defined as follows:

Case 1: If $R_i : a, b, \ldots$, then let $\varphi_i(N, R, \omega) = b$ and $\varphi_j(N, R, \omega) = \mathbb{O} \setminus b$.

Case 2: If $R_i : b, a, \ldots$, then let $\varphi_i(N, R, \omega) = a$ and $\varphi_j(N, R, \omega) = \mathbb{O} \setminus a$.

Case 3: If R_i is such that $T(R_i, \mathbb{O}) \in \mathbb{O} \setminus \{a, b\}$, then let $\varphi_i(N, R, \omega) = T(R_i, \mathbb{O})$ and $\varphi_j(N, R, \omega) = \mathbb{O} \setminus T(R_i, \mathbb{O})$.

Case 4: If R_i is such that $T(R_i, \mathbb{O}) \in \{a, b\}$ and $x \in \mathbb{O} \setminus \{a, b\}$ is the second most preferred object at R_i, then let $\varphi_i(N, R, \omega) = x$ and $\varphi_j(N, R, \omega) = \mathbb{O} \setminus x$.

Note that φ depends only on agent i's preferences. Also, for each preference profile, agent i is assigned only one object.

Case 1: Agent i has an incentive to misrepresent their preferences $R_i' : b, a, \ldots$ and the rule operates under Case 2. Hence, φ is neither *push-up strategy-proof* nor *adjacent strategy-proof*. Note that R_i' is the only misrepresentation that makes agent i better off and starting from R_i, a sequence of $|\mathbb{O}|-1$ drops is required to reach R_i'. Therefore, φ is not manipulable by any sequence of $|\mathbb{O}|-2$ drops of R_i.

Case 2: An analogous argument of Case 1 holds.

Case 3: Agent i has no incentive to misrepresent their preferences since they receive their most preferred object.

Case 4: Without loss of generality, suppose that $a = T(R_i, \mathbb{O})$. Agent i has an incentive to misrepresent that their preferences $R_i' : b, a, \ldots$ and the rule operates under Case 2. Note that R_i' is the only misrepresentation that makes agent i better off and starting from R_i, a sequence of $|\mathbb{O}|-1$ drops is required to reach R_i'. Therefore, φ is not manipulable by any sequence of $|\mathbb{O}|-2$ drops of R_i.

Therefore, when $k = |\mathbb{O}|-2$, φ is *k-drop strategy-proof*. On the other hand, it is neither *push-up strategy-proof* nor *adjacent strategy-proof*. Hence, φ is not *strategy-proof*.

For each pair $k, k' \in \{1, \ldots, |\mathbb{O}|-1\}$ such that $k' < k$, *k-drop strategy-proofness* implies *k'-drop strategy-proofness*. Hence, for each k smaller than $|\mathbb{O}|-1$, *k-drop strategy-proof* does not imply *push-up strategy-proof* or *adjacent strategy-proof*.

Statement 2: *k-push-up strategy-proofness* implies each one of *drop strategy-proofness*, *adjacent strategy-proofness*, and *strategy-proofness* if and only if $k \geq |\mathbb{O}|-1$.

This statement follows from Claim 2 and Example 5.

Claim 2. Each $R_i' \in \mathcal{R}$ is a $|\mathbb{O}|-1$-push-up for R_i.

An analogous argument of that for Claim 1 holds. Hence, we omit the proof of Claim 2. We now show that at least $k = |\mathbb{O}|-1$ is required to have the implication in Statement 2.

Suppose that $k = |\mathbb{O}| - 2$. The following rule is *k-push-up strategy-proof* but not *drop strategy-proof* or *adjacent strategy-proof.*

Example 5. *A rule that is k-push-up strategy-proof where $k = |\mathbb{O}| - 2$, but not drop strategy-proof or adjacent strategy-proof.* Let $(N, R, \omega) \in \mathcal{E}$ and $i, j \in \mathbb{N}$ be such that $N = \{i, j\}$ and $\omega_i \cup \omega_j = \mathbb{O}$. For each $R_i \in \mathcal{R}$, let $a_i \in \mathbb{O}$ be the second least preferred object at R_i in $\mathbb{O}$. Let φ be defined by setting for each $(N, R, \omega) \in \mathcal{E}$,

$$\varphi_i(N, R, \omega) = \mathbb{O} \setminus a_i \text{ and } \varphi_j(N, R, \omega) = a_i.$$

Note that φ depends only on agent i's preferences. For each $R_i \in \mathcal{R}$, the only way for agent i to be made better off is by swapping the preference relation between the second least and the least preferred objects. However, it requires $|\mathbb{O}| - 1$ push-ups of objects. Therefore, when $k = |\mathbb{O}| - 2$, φ is *k-push-up strategy-proof.* On the other hand, it is not *drop strategy-proof* or *adjacent strategy-proof.* Hence, φ is not *strategy-proof.*

For each pair $k, k' \in \{1, \ldots, |\mathbb{O}| - 1\}$ such that $k' < k$, *k-push-up strategy-proofness* implies *k'-push-up strategy-proofness.* Hence, for each smaller k than $|\mathbb{O}| - 1$, *k-push-up strategy-proof* does not imply *drop strategy-proof* or *adjacent strategy-proof.*

Statement 3: *Adjacent strategy-proofness* is equivalent to *strategy-proofness.*

Proof. Suppose that there is $(N, R, \omega) \in \mathcal{E}$, $i \in N$, and $R'_i \in \mathcal{R}$ such that

$$\varphi_i(N, R'_i, R_{-i}, \omega) \; P_i \; \varphi_i(N, R, \omega).$$

Let $K' \in \{1, \ldots, |\mathbb{O}|\}$ and $\{a'_1, \ldots, a'_{K'}\}$ be such that

$$\varphi_i(N, R'_i, R_{-i}, \omega) = \{a'_1, \ldots, a'_{K'}\},$$

and for each $k \in \{1, \ldots, K' - 1\}$, $a'_k \; P_i \; a'_{k+1}$. Let $\tilde{R}_i \in \mathcal{R}$ be such that

$$\tilde{R}_i : a'_1, \ldots, a'_{K'}, \ldots.$$

By *adjacent strategy-proofness*,

$$\varphi_i(N, \tilde{R}_i, R_{-i}, \omega) = \{a'_1, \ldots, a'_{K'}\}. \tag{1}$$

Let $K \in \{1, \ldots, |\mathbb{O}|\}$ and $\{a_1, \ldots, a_K\}$ be such that

$$\varphi_i(N, R, \omega) = \{a_1, \ldots, a_K\},$$

and for each $k \in \{1, \ldots, K-1\}$, $a_k \; P_i \; a_{k+1}$. Note that (1) implies that for each $\hat{R}_i \in \mathcal{R}$ such that $T(\hat{R}_i, \mathbb{O}) = a'_1$,

$$a'_1 \in \varphi_i(N, \hat{R}_i, R_{-i}, \omega).$$

In particular, let $R^1_i \in \mathcal{R}$ be such that $T(R^1_i, \mathbb{O}) = a'_1$ and for each pair $o, o' \in \mathbb{O} \setminus \{a'_1\}$, $oR^1_i o'$ if and only if $oR_i o'$. Then

$$a'_1 \in \varphi_i(N, R^1_i, R_{-i}, \omega).$$

On the other hand, starting from R_i, by *adjacent strategy-proofness*, if $a'_1 \neq a_1$, then

$$a'_1 \notin \varphi_i(N, R^1_i, R_{-i}, \omega),$$

a contradiction. Hence, $a'_1 = a_1$.

Similarly, (1) implies that for each $\hat{R}_i \in \mathcal{R}$ such that $\hat{R}_i : a'_1, a'_2, \ldots$, we have

$$\{a'_1, a'_2\} \subseteq \varphi_i(N, \hat{R}_i, R_{-i}, \omega).$$

In particular, let $R^2_i \in \mathcal{R}$ be such that $R^2_i : a'_1, a'_2, \ldots$, and for each pair $o, o' \in \mathbb{O} \setminus \{a'_1, a'_2\}$, $oR^2_i o'$ if and only if $oR_i o'$. Then

$$\{a'_1, a'_2\} \subseteq \varphi_i(N, R^2_i, R_{-i}, \omega).$$

On the other hand, starting from R_i, by *adjacent strategy-proofness*, if $a'_2 \neq a_2$, then

$$a'_2 \notin \varphi_i(N, R^2_i, R_{-i}, \omega),$$

a contradiction. Hence, $a'_2 = a_2$.

By applying an analogous argument for each $k \in \{3, \ldots, K\}$,

$$a'_k = a_k.$$

Finally, suppose that $K' > K$. Hence,

$$\{a_1, \ldots, a_K\} \subset \{a'_1, \ldots, a'_{K'}\}.$$

Again, (1) implies that for each $\hat{R}_i \in \mathcal{R}$ such that

$$\hat{R}_i : a'_1, \ldots, a'_{K'}, \ldots$$

we have

$$\varphi_i(N, \hat{R}_i, R_{-i}, \omega) = \{a'_1, \ldots, a'_{K'}\}.$$

In particular, let $R_i^{K+1} \in \mathcal{R}$ be such that

$$R_i^{K+1} : a'_1, \ldots, a'_K, a'_{K+1}, \ldots.$$

Note that for each $k \in \{1, \ldots, K\}$, $a'_k = a_k$. Starting from R_i, *adjacent strategy-proofness* implies that

$$a'_{K+1} \notin \varphi_i(N, R_i^{K+1}, R_{-i}, \omega),$$

a contradiction. Hence, $K' = K$. □

B Proofs of Lemmas 1 and 2

Proof of Lemma 1. The proof is by contraposition. Let $(N, R, \omega) \in \mathcal{E}$, $i \in N$, a pair $a, b \in \bigcup \omega_i$, and $R'_i \in \mathcal{R}$ be such that a and b are adjacent at R_i, R'_i is an adjacent swap for R_i, $a \; P_i \; b$, and $b \; P'_i \; a$. Without loss of generality, suppose that (i) there is $o \in SU(R_i, a)$ such that $o \in \varphi_i(N, R, \omega)$ and $o \notin \varphi_i(N, R'_i, R_{-i}, \omega)$, and (ii) for each $o' \in SU(R_i, o)$, $o' \in \varphi_i(N, R, \omega)$ if and only if $o' \in \varphi_i(N, R'_i, R_{-i}, \omega)$. Then

$$\varphi_i(N, R, \omega) \; P'_i \; \varphi_i(N, R'_i, R_{-i}, \omega).$$

Hence, φ is not *adjacent strategy-proof.* □

Proof of Lemma 2. Let φ satisfy the *endowment lower bound* and *splitting invariance.* Let $(N, R, \omega) \in \mathcal{E}$. Let $(N', R', \omega') \in \mathcal{E}$ be such that

- for each $i \in N$, there is $N^i \subset N'$ such that for each $j \in N^i$, $R'_j = R_i$ and $\bigcup_{j \in N'} \omega'_j = \omega_i$ with $|\omega'_j| = 1$,
- for each pair $i, j \in N$, $N^i \cap N^j = \emptyset$, and
- $\bigcup_{i \in N} N^i = N'$.

By the *endowment lower bound*,

$$\text{for each } j \in N',\ \varphi_j(N', R', \omega')\ R'_j\ \omega'_j. \tag{2}$$

By *splitting invariance*,

$$\text{for each } \ i \in N,\ \varphi_i(N, R, \omega) = \bigcup_{j \in N^i} \varphi_j(N', R', \omega'). \tag{3}$$

Then (2) and (3) together imply that for each $i \in N$, there is $\sigma : \omega_i \to \varphi_i(N, R, \omega)$ such that for each $a \in \omega_i$ with $\omega'_j = a$, where $j \in N^i$, $\sigma(a) = \varphi_j(N', R', \omega')$. Thus,

$$\sigma(a)\ R_i\ a.$$

□

C Proofs of The Three Characterizations of GTTC

Lemma 3. GTTC satisfies

- *efficiency*,
- the *strong endowment lower bound*,
- *upper invariance*,
- *drop strategy-proofness*, and
- *splitting invariance*.

Proof of Lemma 3. It is known that GTTC is *efficient* (Fujita et al., 2015). Because each agent exchanges each object they own only with an object they prefer, GTTC meets the *strong endowment lower bound*. Also, it is trivial that GTTC is *splitting invariant*.

Let $(N, R, \omega) \in \mathcal{E}$, $i \in N$, and a pair $a, b \in \bigcup_{j \in N} \omega_j$ be such that a and b are adjacent at R_i. Without loss of generality, suppose that $a \; P_i \; b$. Let $R'_i \in \mathcal{R}$ be an adjacent swap for R_i, and $b \; P'_i \; a$. Let Step k be the step when object a is assigned to an agent when GTTC is applied to (N, R, ω). Then for each $k' < k$, a cycle forms at Step k' when GTTC is applied to (N, R, ω) if and only if the same cycle forms at Step k' when GTTC is applied to $(N, R'_i, R_{-i}, \omega)$. Therefore, GTTC is *upper invariant*.

Finllay, we show that GTTC is *drop strategy-proof*. Suppose that there is $(N, R, \omega) \in \mathcal{E}$, $i \in N$, $b \in \bigcup_{j \in N} \omega_j$, and $R'_i \in \mathcal{R}$ such that R'_i is a drop for R_i at b. Suppose that

$$G_i(N, R'_i, R_{-i}, \omega) \; R_i \; G_i(N, R, \omega). \tag{4}$$

Let $m \equiv |\omega_i|$. Let $\{a_1, \ldots, a_m\} \subseteq \bigcup_{j \in N} \omega_j$ be such that for each $k \in \{1, \ldots, m\}$, $a_k \in G_i(N, R, \omega)$, and for each $k \in \{1, \ldots, m-1\}$ and $a_k \; P_i \; a_{k+1}$. For each $k \in \{1, \ldots, m\}$, let Step s_k be the step when agent i receives a_k when GTTC is applied to (N, R, ω).

We first show that for each $o \in U(R_i, a_1)$,

$$o \in G_i(N, R, \omega) \iff o \in G_i(N, R'_i, R_{-i}, \omega).$$

Note that agent i does not belong to any cycle until Step s_1 when GTTC is applied to either (N, R, ω) or $(N, R'_i, R_{-i}, \omega)$. Hence, for each $o \in \bigcup_{j \in N} \omega_j$ such that $o \in SU(R_i, a_1)$,

$$o \notin G_i(N, R, \omega) \text{ and } o \notin G_i(N, R'_i, R_{-i}, \omega).$$

By (4),

$$a_1 \in G_i(N, R, \omega) \text{ and } a_1 \in G_i(N, R'_i, R_{-i}, \omega).$$

Let $k' \in \{1, \ldots, m-1\}$. Suppose that for each $o \in U(R_i, a_{k'})$,

$$o \in G_i(N, R, \omega) \iff o \in G_i(N, R'_i, R_{-i}, \omega).$$

Now we show that for each $o \in U(R_i, a_{k'+1})$,

$$o \in G_i(N, R, \omega) \iff o \in G_i(N, R'_i, R_{-i}, \omega).$$

By the induction hypothesis we focus on objects in $SL(R_i, o_{k'}) \cap U(R_i, o_{k'+1})$. Suppose that $b \in U(R_i, o_{k'})$. Then either (i) there is $j \neq i$ such that agent j receives b or (ii) agent i owns and receives b. In either case, for each $s_{k'} < s \leq s_{k'+1}$, a cycle forms at Step s when GTTC is applied to (N, R, ω) if and only if the same cycle forms when GTTC is applied to $(N, R'_i, R_{-i}, \omega)$. Hence,

$$G_i(N, R, \omega) = G_i(N, R'_i, R_{-i}, \omega).$$

Suppose that $b \in SL(R_i, o_{k'+1})$. For each $s_{k'} < s \leq s_{k'+1}$, a cycle forms at Step s when GTTC is applied to (N, R, ω) if and only if the same cycle forms when GTTC is applied to $(N, R'_i, R_{-i}, \omega)$. Suppose that $b \in SL(R_i, o_{k'}) \cap U(R_i, o_{k'+1})$. For each $s_{k'} < s < s_{k'+1}$, agent i does not belong to any cycle at Step s when GTTC is applied to either (N, R, ω) or $(N, R'_i, R_{-i}, \omega)$. Hence, for each $o \in SL(R_i, a_{k'}) \cap SU(R_i, a_{k'})$,

$$o \notin G_i(N, R, \omega) \text{ and } o \notin G_i(N, R'_i, R_{-i}, \omega).$$

By (4),

$$a_{k'+1} \in G_i(N, R, \omega) \text{ and } a_{k'+1} \in G_i(N, R'_i, R_{-i}, \omega).$$

□

Proof of Theorem 1. Let φ satisfy all properties in Theorem 1, and $(N, R, \omega) \in \mathcal{E}$. Let A be the algorithm associated with G. Define a variant A' that is identical to A, except that only one cycle is assigned at each step. That is, at Step 1 in the algorithm for A, if there are k cycles, then order these cycles $c_1, \ldots, c_k$. In algorithm A', at Step 1, agents point similarly as in A and the same cycles form, but assign only cycle c_1. Then, at Step 2,..., Step k of A', cycles $c_2, \ldots, c_k$ remain, and assign them in order. Step 2 of A coincides with Step $k+1$ of A', and we repeat the process. Clearly A and A' result in the same allocation. Hereafter, assume that we are using A', and there is only one cycle per step.

Consider Step 1 of the algorithm for G at (N, R, ω). Let $j_1, a_1, j_2, \ldots, j_m, a_m, j_1$ be the occurring cycle, where $j_1, \ldots, j_m \in N$, $a_x \in \omega_{j_{x+1}} \mod m$, and j_x points to a_x. Construct $R^1 \in \mathcal{R}^N$ as follows: for each $i \in N$,

- (Preserve ω_i order) for each $b, c \in \omega_i$, $b \; R_i \; c \Leftrightarrow b \; R_i^1 \; c$,
- (i in the cycle) if $i = j_x$ for some $x \in \{1, \ldots, m\}$, then
  - (Top rank cycled object) a_x is top-ranked in R_i^1, and
  - (All else ranked lower) for each $b \in \omega_i$ and each $c \in \bigcup_{j \in N} \omega_j \setminus (\omega_i \cup a_x)$, $b \; R_i^1 \; c$,
- (i not in the cycle) if $i \notin \{j_1, \ldots, j_x\}$, then for each $b \in \omega_i$, and each $c \in \bigcup_{j \in N \setminus \{i\}} \omega_j$, $b \; R_i^1 \; c$.

The intuition for R^1 is that for each agent we drop all objects not in their endowment; if they happen to be in a cycle at Step 1, then we keep their first-ranked object at the top. Throughout the remainder of the proof, N and ω are fixed. Abusing notation, for any R, write $\varphi(R) = \varphi(N, R, \omega)$.

Claim 3. $\varphi(R^1) = G(R^1)$.

Proof. By the *strong endowment lower bound*, for each $i \in N$ such that i is not in a cycle at Step 1, $\varphi_i(R^1) = G_i(R^1) = \omega_i$. If $a_1 \in \omega_{j_1}$ (j_1 cycles with their own object), then by the *strong endowment lower bound* $\varphi_{j_1}(R^1) = G_{j_1}(R^1) = \omega_{j_1}$ and we are done. Let $a_1 \notin \omega_{j_1}$. If $a_1 \notin \varphi_{j_1}(R^1)$, then by the *strong endowment lower bound*, $\varphi_{j_1}(R^1) = \omega_{j_1}$. This implies similarly that φ assigns each other agent in the cycle their endowment—violating *efficiency*. If $a_1 \in \varphi_{j_1}(R^1)$, then by the *strong endowment lower bound*, $a_2 \in \varphi_{j_2}(R^1)$, and similarly so for each agent in the cycle. Thus, $\varphi(R^1) = G(R^1)$. $\square$

Note that this holds for arbitrary ordering of objects below agents' endowments.

We recursively define a family $\mathcal{Q}(\cdot)$ of subsets of $\mathcal{R}^N$. The proof proceeds by induction on this family. The intuition is that starting from R^1, for one agent i we "add back in" an object that we had to drop to construct R_i^1. Repeating this, we eventually arrive at a family that contains R. There are many paths to R, each with a different sequence of objects used. Importantly, each $\mathcal{Q}(k+1)$ is the *set* of preference profiles with each element featuring an agent/object pair chosen to implement the "add back in" procedure, starting from *some* preference profile in $\mathcal{Q}(k)$.

We now proceed formally. Recall that the construction of R^1 did not generally pin down a unique preference profile e.g. objects ranked below an agent's endowment are arbitrarily ranked. Let $\mathcal{Q}(1) \subset \mathcal{R}^N$ be the set of preference profiles that can be constructed as R^1. For each $k \geq 1$, let $\mathcal{Q}(k+1)$ be the set of $R^{k+1} \in \mathcal{R}^N$ such that there is $i \in N$, $x \in \bigcup_{j\in N} \omega_j$, and $R^k \in \mathcal{Q}(k)$ where

- (Only i changes) $R^{k+1}_{-i} = R^k_{-i}$,
- (Add back in highest-ranked) for each $a' \in \left\{a \in \bigcup_{j\in N} \omega_j : \ \forall b \in \omega_i, \ b \ P^k_i \ a\right\}$, we have that $x \ R_i \ a'$, and
- (New object rank consistent with R_i) for each $a \in \bigcup_{j\in N} \omega_j$, $a \ R_i \ x \Leftrightarrow a \ R^{k+1}_i \ x$.

By finiteness of (N, R, ω), for some $\bar{k}$, $R \in \mathcal{Q}(\bar{k})$.

Induction Hypothesis: Let $k < \bar{k}$. Assume that for each $R^k \in \mathcal{Q}(k)$, $\varphi(N, R^k, \omega) = G(N, R^k, \omega)$. By Claim 3, the statement holds for $k = 1$.

Induction Step: Let $R^{k+1} \in \mathcal{Q}(k+1)$, and consider $G(R^{k+1})$. Let $c_1, \dots, c_{\bar{s}}$ be the cycles that occur in the algorithm for G at R^{k+1}, where if $s < s'$, then c_s occurs in a step before $c_{s'}$. Let $c_s = \{\{i_{s,1}, \dots, i_{s,m(s)}\}, \{a_{s,1}, \dots, a_{s,m(s)}\}\}$ where $i_{s,1}, \dots, i_{s,m(s)} \in N$ is the set of agents in the cycle and $a_{s,1}, \dots, a_{s,m(s)} \in \bigcup_{j\in N} \omega_j$ be the respective objects that each agent points to / is assigned in the cycle. For example, at Step s of the algorithm for G at R^{k+1}, $i_{s,1}$ points to $a_{s,1} \in \omega_{i_{s,2}}$ points to $i_{s,2}$ points to $a_{s,2} \in \omega_{i_{s,3}}$ and so on.

We will show that $\varphi(R^{k+1}) = G(R^{k+1})$ by induction on the set of cycles. Claim 4 shows that the first cycle in the algorithm for G at R^{k+1} is assigned in $\varphi(R^{k+1})$. The subsequent claim is the induction step.

Claim 4. For each $\ell \in \{1, \dots, m(1)\}$, $a_{1,\ell} \in \varphi_{i_{1,\ell}}(R^{k+1})$.

Proof. For each $\ell \in \{1, \dots, m(1)\}$, let $L_{1,\ell} \subseteq \bigcup_{j\in N} \omega_j \setminus \omega_{i_{1,\ell}}$ be the set of objects ranked below $a_{1,\ell}$ but above some object in $\omega_{i_{1,\ell}}$ according to $R^{k+1}_{i_{1,\ell}}$. This indicates that objects in $L_{1,\ell}$ were "added back in" along the process.

If for each $\ell \in \{1, \dots, m(1)\}$, $L_{1,\ell} = \emptyset$, then by *efficiency* and the *strong endowment lower bound*, we have the desired result.

Let there be $\ell \in \{1, \ldots, m(1)\}$ such that $L_{1,\ell} \neq \emptyset$. Let ℓ be such an index. Consider the preference profile where $i_{1,\ell}$ drops an object in $L_{1,\ell}$, and each other agent is the same. This preference profile is in $\mathcal{Q}(k)$ and by the induction hypothesis, φ coincides with G for profiles in $\mathcal{Q}(k)$. Since c_1 also occurs in the algorithm for G at this preference profile, by *drop strategy-proofness*, $a_{1,\ell} \in \varphi_{i_{1,\ell}}(R^{k+1})$. Consider $i_{1,\ell+1}$. If $L_{1,\ell+1} \neq \emptyset$, then we can repeat this reasoning. Let $L_{1,\ell+1} = \emptyset$. By definition of c_1, $a_{1,\ell} \in \omega_{i_{1,\ell+1}}$. Since $a_{1,\ell} \notin \varphi_{i_{1,\ell+1}}(R^{k+1})$, by the *strong endowment lower bound* and the fact that $a_{1,\ell+1}$ is the only object ranked above $\omega_{i_{1,\ell}}$ in $R^{k+1}_{i_{1,\ell+1}}$, $a_{1,\ell+1} \in \varphi_{i_{1,\ell+1}}(R^{k+1})$. Repeating this reasoning, we have the desired result. □

Let $s < \bar{s}$, and assume that $c_1, \ldots, c_s$ is assigned under $\varphi(R^{k+1})$. That is, for each $i \in N$,

$$\bigcup_{s' \leq s} \left(\bigcup_{\ell:\ i_{s',\ell} = i} a_{s',\ell} \right) \subseteq \varphi_i(R^{k+1}).$$

We will refer to this as the Cycle Induction Hypothesis, to differentiate from the previous Induction Hypothesis.

Consider cycle c_{s+1}. For each $\ell \in \{1, \ldots, m(s+1)\}$, construct $L_{s+1,\ell}$ as before: let $L_{s+1,\ell} \subseteq \bigcup_{j \in N} \omega_j \setminus \omega_{i_{s+1,\ell}}$ be the set of objects ranked strictly below $a_{s+1,\ell}$ at $R^{k+1}_{i_{s+1,\ell}}$, and strictly above some object in $\omega_{i_{s+1,\ell}}$ at $R^{k+1}_{i_{s+1,\ell}}$. The following claim states that cycle c_{s+1} in the algorithm for G at R^{k+1} is assigned in $\varphi(R^{k+1})$.

Claim 5. For each $\ell \in \{1, \ldots, m(s+1)\}$, $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell}}(R^{k+1})$.

Proof. We divide into two cases.

Case 1: There is $\ell \in \{1, \ldots, m(s+1)\}$ such that $L_{s+1,\ell} \neq \emptyset$. Let ℓ satisfy this condition. Note that if any agent drops an object, then we are in an economy in $\mathcal{Q}(k)$, and by the Induction Hypothesis, φ coincides with G. Consider agent $i_{s+1,\ell}$ and a drop of some object in $L_{s+1,\ell}$. By *drop strategy-proofness*,

$$\bigcup_{s' \leq s+1} \left(\bigcup_{\ell':\ i_{s',\ell'} = i_{s+1,\ell}} a_{s',\ell'} \right) \subseteq \varphi_{i_{s+1,\ell}}(R^{k+1}).$$

Now consider agent $i_{s+1,\ell+1}$. If $L_{s+1,\ell+1} \neq \emptyset$, then the same reasoning holds. Let $L_{s+1,\ell+1} = \emptyset$.

If $i_{s+1,\ell+1}$ did not appear in any cycle in $\{c_1, \ldots, c_s\}$, then they have not been assigned any objects in $SU(R^{k+1}_{i_{s+1,\ell+1}}, a_{s+1,\ell+1})$, and by the *strong endowment lower bound* and $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell}}(R^{k+1}) \cap \omega_{i_{s+1,\ell+1}}$, we have that $a_{s+1,\ell+1} \in \varphi_{i_{s+1,\ell+1}}(R^{k+1})$.

Let $i_{s+1,\ell+1}$ appear in some cycle in $\{c_1, \ldots, c_s\}$. By the Cycle Induction Hypothesis, $\varphi(R^{k+1})$ assigns the agent all the objects that they point to for each cycle that they appear in, and none of the others in the cycles:

$$\bigcup_{s' \leq s} \left(\bigcup_{\ell' \colon i_{s',\ell'} = i_{s+1,\ell+1}} a_{s',\ell'} \right) \subseteq \varphi_{i_{s+1,\ell+1}}(R^{k+1}),$$

and,

$$\left(\bigcup_{s' \leq s} \left(\bigcup_{\ell' \colon i_{s',\ell'} \neq i_{s+1,\ell+1}} a_{s',\ell'} \right) \right) \cap \varphi_{i_{s+1,\ell+1}}(R^{k+1}) = \emptyset.$$

Since $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell}}(R^{k+1}) \cap \omega_{i_{s+1,\ell+1}}$ and $L_{s+1,\ell+1} = \emptyset$, by the *strong endowment lower bound*, $a_{s+1,\ell+1} \in \varphi_{i_{s+1,\ell+1}}(R^{k+1})$.

Case 2: For each $\ell \in \{1, \ldots, m(s+1)\}$, $L_{s+1,\ell} = \emptyset$. If there is $\ell \in \{1, \ldots, m(s+1)\}$ such that $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell}}(R^{k+1})$, then the same reasoning as in Case 1 holds. If there is $j \in N \setminus \{i_{s+1,1}, \ldots, i_{s+1,m(s+1)}\}$ such that $a_{s+1,\ell} \in \varphi_j(R^{k+1})$, then the same reasoning as in Case 1 occurs starting for agents $i_{s+1,\ell+1}, \ldots, i_{s+1,\ell-1}$. At agent $i_{s+1,\ell}$, the reasoning above shows that $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell}}(R^{k+1})$, violating feasibility. If the two above cases are not true, then by the Cycle Induction Hypothesis and the *strong endowment lower bound*, for each $\ell \in \{1, \ldots, m(s+1)\}$, $a_{s+1,\ell} \in \varphi_{i_{s+1,\ell+1}}(R^{k+1}) \cap \omega_{i_{s+1,\ell+1}}$, and we have a violation of *efficiency*. □

Since there are $\bar{s} < \infty$ number of cycles in the algorithm for G at R^{k+1}, we showed by induction that for each $i \in N$,

$$\bigcup_{s' \leq \bar{s}} \left(\bigcup_{\ell' \colon i_{s',\ell'} = i_{s+1,\ell}} a_{s',\ell'} \right) \subseteq \varphi_{i_{s+1,\ell}}(R^{k+1}).$$

Since

$$\bigcup_{s' \leq \bar{s}} \{a_{s',1}, \ldots, a_{s',m(s')}\} = \bigcup_{j \in N} \omega_j,$$

the line above holds with equality, and we have the desired result. This completes the proof of Theorem 1. □

Proof of Theorem 2. The proof is similar to Theorem 1, and we explain points of divergence. Since the only difference between the two theorems is the replacement of *drop strategy-proofness* with *upper invariance*, we focus on parts where the latter can be used to prove the same statements where the former is invoked.

The first time *drop strategy-proofness* is used is in Claim 4. Consider when, starting from $R_{i_{1,\ell}}$, $i_{1,\ell}$ drops an object in $L_{1,\ell}$ to arrive at a manipulation $R'_{i_{1,\ell}}$. Let a^* be this object. Note that starting from $R_{i_{1,\ell}}$ this manipulation can be achieved by a sequence of adjacent swaps in the order of objects i.e. swap objects a^* and the object directly below it, then swap objects a^* and the object subsequently below it, ...until a^* is at the bottom. Suppose by contradiction that $a_{1,\ell} \notin \varphi(R^{k+1})$. At the first swap, notice that *upper invariance* implies that $i_{1,\ell}$ is not assigned $a_{1,\ell}$ by φ at the subsequent preference profile. Similarly, $i_{1,\ell}$ is not assigned $a_{1,\ell}$ by φ for each subsequent preference profile where $i_{1,\ell}$ swaps a^* for the next lowest object. Consider $\varphi(R'_{i_{1,\ell}}, R_{-i_{1,\ell}})$. Since this preference profile is in $Q(k)$, φ coincides with G at this profile. In G, c_1 occurs and $i_{1,\ell}$ is assigned $a_{i_{1,\ell}}$, a contradiction.

The same reasoning holds for each subsequent agent in c_1, and so Claim 4 holds when invoking *upper invariance* as opposed to *drop strategy-proofness*.

The next time *drop strategy-proofness* is used is in Claim 5. Consider $i_{s+1,\ell}$ in Case 1. The same reasoning as in the previous two paragraphs shows the same implication by *drop strategy-proofness*. Thus, Claim 5 holds when invoking *upper invariance* as opposed to *drop strategy-proofness*. □

Proof of Theorem 3. Let $\varphi \neq G$ satisfy *efficiency* and the *endowment lower bound*.

Claim 6. For each $(N, R, \omega) \in \mathcal{E}$ such that for each $i \in N$, $|\omega_i| = 1$, if $\varphi(N, R, \omega) \neq G(N, R, \omega)$, there is $i \in N$ such that

$$G_i(N, R, \omega) \; P_i \; \varphi_i(N, R, \omega) \; P_i \; \omega_i.$$

Proof. Let $(N, R, \omega) \in \mathcal{E}$ be such that for each $i \in N$, $|\omega_i| = 1$, and $\varphi(N, R, \omega) \neq G(N, R, \omega)$. Since GTTC is *efficient*, there is $i \in N$ such that $G_i(N, R, \omega) \; P_i \; \varphi_i(N, R, \omega)$. Suppose that for each $i \in N$ with $G_i(N, R, \omega) \; P_i \; \varphi_i(N, R, \omega)$, we have $\omega_i \; R_i \; \varphi_i(N, R, \omega)$. Since φ satisfies the *endowment lower bound*, for each such i, $\omega_i = \varphi_i(N, R, \omega)$. Let

$$N_1 = \{i \in N : \varphi_i(N, R, \omega) \; P_i \; G_i(N, R, \omega)\}$$
$$N_2 = \{i \in N : \varphi_i(N, R, \omega) = G_i(N, R, \omega)\}.$$

Since for each $i \in N$ with $G_i(N, R, \omega) \; P_i \; \varphi_i(N, R, \omega)$, we have $\omega_i = \varphi_i(N, R, \omega)$,

$$\bigcup_{i \in N_1 \, \cup \, N_2} \varphi_i(N, R, \omega) = \bigcup_{i \in N_1 \, \cup \, N_2} \omega_i.$$

Let $x \in \mathcal{X}(N, R, \omega)$ be such that for each $i \in N_1 \, \cup \, N_2$, $x_i = \varphi_i(N, R, \omega)$. Then,

- $\bigcup_{i \in N_1 \, \cup \, N_2} x_i = \bigcup_{i \in N_1 \, \cup \, N_2} \omega_i$,
- for each $i \in N_1$, $x_i \; P_i \; G_i(N, R, \omega)$,
- for each $i \in N_2$, $x_i \; R_i \; G_i(N, R, \omega)$.

This contradicts the fact that $G(N, R, \omega)$ is in the core of (N, R, ω). $\square$

Let $(N, R, \omega) \in \mathcal{E}$ be such that for each $i \in N$, $|\omega_i| = 1$. For each $i \in N$, let $R'_i \in \mathcal{R}$ be such that

- for each pair $a, b \in \bigcup_{j \in N \setminus \{i\}} \omega_j$, $a \; R'_i \; b$ if and only if $a \; R_i \; b$,
- $G_i(N, R, \omega) \; R'_i \; \omega_i$, and
- for each $a \in \bigcup_{j \in N \setminus \{i\}} \omega_j$, if $G_i(N, R, \omega) \; P_i \; a$, then $\omega_i \; P'_i \; a$.

Note that for each $N' \subseteq N$, $G(N, R'_{N'}, R_{-N'}, \omega) = G(N, R, \omega)$.

Claim 7. $\varphi(N, R', \omega) = G(N, R', \omega)$.

Proof. Because φ meets the *endowment lower bound*, for each $i \in N$, $|\varphi_i(N, R', \omega)| = 1$. Let $S \in \{1, \ldots, |N|\}$. For each $s \in \{1, \ldots, S\}$, let $N_s \subseteq N$ be such that for each $i \in N_s$, there is a cycle that contains agent i at Step s when GTTC is applied to (N, R', ω). Also, for each $s \in \{1, \ldots, S\}$, let $O_s \subseteq \bigcup_{i \in N} \omega_i$ be such that each $a \in O_s$ is assigned to an agent at Step s when GTTC is applied to (N, R', ω). The proof is by induction on s.

Let $s = 1$. For each $i \in N_1$, $U(R_i, \omega_i) \cap \bigcup_{j \in N} \omega_j = \{G_i(N, R', \omega), \omega_i\}$. Because φ meets the *endowment lower bound*, for each $i \in N_1$,

$$\varphi_i(N, R', \omega) \in \{G_i(N, R', \omega), \omega_i\}.$$

Let $M_1 \subseteq N_1$ be such that the agents in M_1 form a cycle. Suppose that there is $i \in M_1$ such that $\varphi_i(N, R', \omega) = \omega_i$. Then for each $i \in M_1$, $\varphi_i(N, R', \omega) = \omega_i$. There is $x \in \mathcal{X}(N, R', \omega)$ such that for each $i \in M_1$, $x_i = G_i(N, R', \omega)$, and x Pareto-dominates $\varphi(N, R', \omega)$, contradicting the fact that φ is *efficient*. Hence, for each $i \in N_1$, $\varphi_i(N, R', \omega) = G_i(N, R', \omega)$.

Induction Hypothesis: Let $s' \in \{2, \ldots, S\}$. Assume that for each $s < s'$ and each $i \in N_s$, $\varphi_i(N, R', \omega) = G_i(N, R', \omega)$.

Induction Step: Let $s = s'$. Because φ meets the *endowment lower bound*, by the induction hypothesis, for each $i \in N_{s'}$,

$$\varphi_i(N, R', \omega) \in \{G_i(N, R', \omega), \omega_i\}.$$

Let $M_{s'} \subseteq N_{s'}$ be such that the agents in $M_{s'}$ form a cycle. Suppose that there is $i \in M_{s'}$ such that $\varphi_i(N, R', \omega) = \omega_i$. Then for each $i \in M_{s'}$, $\varphi_i(N, R', \omega) = \omega_i$. There is $y \in \mathcal{X}(N, R', \omega)$ such that for each $i \in M_{s'}$, $x_i = G_i(N, R', \omega)$, and y Pareto-dominates $\varphi(N, R', \omega)$, contradicting the fact that φ is *efficient*. Hence, for each $i \in N_s$, $\varphi_i(N, R', \omega) = G_i(N, R', \omega)$. $\square$

In addition to *efficiency* and the *endowment lower bound*, let φ also be *upper invariant*. We show that for each $(N, R, \omega) \in \mathcal{E}$ such that for each $i \in N$, $|\omega_i| = 1$,

$$\varphi(N, R, \omega) = G(N, R, \omega).$$

Let $(N, R, \omega) \in \mathcal{E}$ be such that for each $i \in N$, $|\omega_i| = 1$. Let $N' \subseteq N$. Let $\tilde{R} \in \mathcal{R}^{\mathbb{N}}$ be such that for each $i \in N'$, $\tilde{R}_i = R_i$, and for each $i \in N \backslash N'$, $\tilde{R}_i = R'_i$. The proof is by induction on $|N'|$.

Induction Hypothesis: Let $n \in \{1, \ldots, |N|\}$. Assume that if $|N'| < n$, $\varphi(N, \tilde{R}, \omega) = G(N, \tilde{R}, \omega)$. By Claim 7, the statement holds for $|N'| = 0$.

Induction Step: Let $|N'| = n$. Suppose that $\varphi(N, \tilde{R}, \omega) \neq G(N, \tilde{R}, \omega)$. By Claim 6, there is $i \in N$ such that

$$G_i(N, \tilde{R}, \omega) \ \tilde{P}_i \ \varphi_i(N, \tilde{R}, \omega) \ \tilde{P}_i \ \omega_i.$$

Let $i \in N$ be such an agent. Note that $i \in N'$, otherwise it contradicts with the definition of R'_i. By the induction hypothesis,

$$\varphi_i(N, R'_i, \tilde{R}_{-i}, \omega) = G_i(N, R'_i, \tilde{R}_{-i}, \omega).$$

Note that $G_i(N, R'_i, \tilde{R}_{-i}, \omega) \in SU(R'_i, \omega_i)$. Because φ is *upper invariant*,

$$\varphi_i(N, \tilde{R}, \omega) = \varphi_i(N, R'_i, \tilde{R}_{-i}, \omega).$$

Therefore,

$$\varphi(N, \tilde{R}, \omega) = G(N, \tilde{R}, \omega).$$

Finally, let φ also be *splitting invariant*. Let (N, R, ω) be such that there is $i \in N$ such that $|\omega_i| > 1$. Let $(N', R', \omega') \in \mathcal{E}$ be such that

- for each $i \in N$, there is $N^i \subset N'$ such that for each $j \in N^i$, $R'_j = R_i$, and $\bigcup \omega'_j = \omega_i$,
- for each pair $i, j \in N$, $N^i \cap N^j = \emptyset$, and
- $\bigcup N^i = N'$.

Note that

$$\varphi(N', R', \omega') = G(N', R', \omega').$$

Because GTTC is *splitting invariant*, for each $i \in N$,

$$G_i(N, R, \omega) = \bigcup_{j \in N^i} G_j(N', R', \omega').$$

Because φ is *splitting invariant*, for each $i \in N$,

$$\varphi_i(N, R, \omega) = \bigcup_{j \in N^i} \varphi_j(N', R', \omega').$$

Therefore,

$$\varphi(N, R, \omega) = G(N, R, \omega).$$

$\square$

D Independence of Axioms of Theorems 1, 2, and 3

We first define two well-known families of rules:

No Trade Rule For each $(N, R, \omega) \in \mathcal{E}$,

$$NT(N, R, \omega) = \omega.$$

Dictatorship For each $N \subseteq \mathbb{N}$, there is $i \in N$ such that for each $(N, R, \omega) \in \mathcal{E}$,

$$D_i(N, R, \omega) = \bigcup_{i \in N} \omega_i.$$

Independence of Axioms

Theorem 1

- The No Trade rule satisfies all properties in Theorem 1 except for *efficiency*.
- The Dictatorship satisfies all properties in Theorem 1 except for the *strong endowment lower bound*.

- The following rule satisfies all properties in Theorem 1 except for *drop strategy-proofness.*

  Let $(N, R, \omega) \in \mathcal{E}$.

  - If $|N| = 4$, denoting by $N = \{i, j, k, l\}$, $(\omega_i, \omega_j, \omega_k, \omega_l) = (b, a, d, c)$, and

    $$R_i : a,\ c,\ b$$
    $$R_j : b,\ d,\ a$$
    $$R_k : a,\ c,\ d$$
    $$R_l : b,\ d,\ c,$$

    then

    $$\varphi(N, R, \omega) = (c, d, a, b),$$

  - otherwise, $\varphi(N, R, \omega) = G(N, R, \omega)$.

  Because GTTC satisfies *efficiency* and the *strong endowment lower bound*, so does φ.

  Let $(N, R, \omega) \in \mathcal{E}$ satisfy the conditions in the first bullet point. Let $R'_i \in \mathcal{R}$ be such that R'_i is a drop for R_i and

  $$R'_i : a,\ b,\ c.$$

  Then

  $$\varphi(N, R'_i, R_{-i}, \omega) = (a, b, c, d).$$

  Because agent i prefers a to c at R_i, φ is not *drop strategy-proof.*

Theorem 2

- The No Trade rule satisfies all properties in Theorem 2 except for *efficiency.*

- The Dictatorship satisfies all properties in Theorem 2 except for the *strong endowment lower bound.*

- The following rule satisfies all properties in Theorem 2 except for *upper invariance*.

  Let $(N, R, \omega) \in \mathcal{E}$.

  - If $|N| = 2$, denoting by $N = \{i, j\}$, $\omega = (a, bc)$, and

$$R_i : b,\ c,\ a$$

$$R_j : a,\ b,\ c,$$

    then

$$\varphi(N, R, \omega) = (c, ab),$$

  - otherwise, $\varphi(N, R, \omega) = G(N, R, \omega)$.

  Because GTTC satisfies *efficiency* and the *strong endowment lower bound*, so does φ.

  Let $(N, R, \omega) \in \mathcal{E}$ satisfy the conditions in the first bullet point. Let $R'_i \in \mathcal{R}$ be such that R'_i is an adjacent swap for R_i at c and a, i.e.,

$$R'_i : b,\ a,\ c.$$

  Then

$$\varphi_i(N, R'_i, R_{-i}, \omega) = b.$$

  Because agent i receives b at R'_i but not at R_i, φ is not *upper invariant*.

Theorem 3

- The No Trade rule satisfies all properties in Theorem 3 except for *efficiency*.

- The following rule satisfies all properties in Theorem 3 except for the *endowment lower bound*.

  Let $\gamma : \{1, \dots, |\mathbb{O}|\} \to \mathbb{O}$ be a bijection. For each $O \subseteq \mathbb{O}$, let γ_O be the order on O according to γ. However, unless it is confusing, we also denote by γ the order on O. Let $(N, R, \omega) \in \mathcal{E}$.

Let $O^1 = O$.

Step *s* Let $T(R_{\omega(\gamma(s))}, O^s) \in \varphi_{\omega(\gamma(s))}(N, R, \omega)$.

Let $O^{s+1} = O^s \setminus \{T(\omega(\gamma(s)), O^s)\}$.

Hence, at each Step s, the owner of s^{th} ranked object receives their most preferred object among the available ones. Because φ is a sequential priority rule, it is *efficient*. Because at each step, who receives an object at that step only depends on the order on the object set, the rule is *upper invariant* and *splitting invariant*. On the other hand, an allocation selected by φ is independent of an endowment profile. Hence, φ does not meet the *endowment lower bound*.

- The following rule satisfies all properties in Theorem 3 except for *upper invariance*.

  Let $a, b, c \in \mathbb{O}$. Let $(N, R, \omega) \in \mathcal{E}$.

  - If $|N| = 3$, denoting by $N = \{i, j, k\}$, $\omega = (a, b, c)$, and

    $$R_i : b,\ c,\ a$$

    $$R_j : a,\ c,\ b$$

    $$R_k : b,\ a,\ c,$$

    then

    $$\varphi(N, R, \omega) = (c, a, b),$$

  - otherwise, $\varphi(N, R, \omega) = G(N, R, \omega)$.

  Clearly, φ is *splitting invariant*. Because GTTC is *efficient*, so is φ. Also, because GTTC meets the *endowment lower bound*, so does φ.

  Let $(N, R, \omega) \in \mathcal{E}$ satisfy the conditions in the first bullet point. Let $R'_i \in \mathcal{R}$ be such that R'_i is an adjacent swap for R_i at c and a, i.e.,

  $$R'_i : \ b,\ a,\ c.$$

Then

$$\varphi_i(N, R'_i, R_{-i}, \omega) = b.$$

Agent i receives b at R'_i but not at R_i. Hence, φ is not *upper invariant.*

- The following rule satisfies all properties in Theorem 3 except for *splitting invariance.*

  Let $\sigma : \{1, \ldots, |\mathbb{N}|\} \to \mathbb{N}$ be a bijection. For each $N \subseteq \mathbb{N}$, let σ_N be the order on N according to σ. However, unless it is confusing, we also denote by σ the order on N. Let $(N, R, \omega) \in \mathcal{E}$.

  - If $|N| = 2$, $|\bigcup_{i \in N} \omega_i| = 3$, and there is $a \in \omega_{\sigma(2)}$ such that

    1. $a \; P_{\sigma(1)} \; \omega_{\sigma(1)}$,
    2. $\left(\bigcup_{i \in N} \omega_i \setminus \{a\}\right) \; R_{\sigma(2)} \; \omega_{\sigma(2)}$, and
    3. for each $b \in \omega_{\sigma(2)}$, $b \; R_{\sigma(2)} \; a$,

    then

    $$(\varphi_{\sigma(1)}(N, R, \omega), \varphi_{\sigma(2)}(N, R, \omega)) = (a, \bigcup_{i \in N} \omega_i \setminus \{a\}),$$

  - otherwise, $\varphi(N, R, \omega) = G(N, R, \omega)$.

  Consider an economy where the conditions in the first bullet point holds. Note that agent $\sigma(1)$ owns one object and agent $\sigma(2)$ owns two objects. Without loss of generality, suppose that agent $\sigma(1)$ owns c and agent $\sigma(2)$ owns a and b. Also, suppose that agent $\sigma(1)$ prefers a to c and agent $\sigma(2)$ prefers each of b and c to a. Then

  $$(\varphi_{\sigma(1)}(N, R, \omega), \varphi_{\sigma(2)}(N, R, \omega)) = (a, bc).$$

  Consider an allocation such that agent $\sigma(1)$ receives more than one object. Then agent $\sigma(2)$'s welfare at that allocation is lower than that of receiving bc. Consider an allocation such that agent $\sigma(1)$ receives b. Again agent $\sigma(2)$'s welfare at that allocation is lower than that of receiving bc. Finally, consider an allocation such that $\sigma(1)$ receives c. Then agent $\sigma(1)$'s welfare at that allocation is lower than that of receiving a. Hence, φ is *efficient.* It is clear that φ meets the *endowment lower* *bound.* Suppose that $R'_{\sigma(1)} \in \mathcal{R}$ is an adjacent swap for $R_{\sigma(1)}$ such that their second and third ranked objects are reversed. So agent $\sigma(1)$'s most preferred objects at $R_{\sigma(1)}$ and $R'_{\sigma(1)}$ are the same. Whether agent $\sigma(1)$ receives the most preferred object at $R_{\sigma(1)}$ and $R'_{\sigma(1)}$ only depends on agent $\sigma(2)$'s preferences. Symmetrically, suppose that $R'_{\sigma(2)} \in \mathcal{R}$ is an adjacent swap for $R_{\sigma(2)}$ such that their second and third ranked objects are reversed. Whether agent $\sigma(2)$ receives their most preferred object at $R_{\sigma(2)}$ and $R'_{\sigma(2)}$ only depends on agent $\sigma(1)$'s preferences. Hence, φ is *upper invariant.*

Let $(N, R, \omega) \in \mathcal{E}$ be satisfy the conditions in the first bullet point. Let $(N', R', \omega') \in \mathcal{E}$ be such that $N' = \{1, 2, 3\}$, $\omega' = (c, b, a)$, and

$$R'_1 : b,\ a,\ c$$

$$R'_2 : c,\ b,\ a$$

$$R'_3 : c,\ b,\ a.$$

Then

$$\varphi(N', R', \omega') = (b, c, a).$$

Because $\varphi_{\sigma(2)}(N, R, \omega) \neq (\varphi_2(N', R', \omega') \cup \varphi_3(N', R', \omega'))$, φ is not *splitting invariant.*

Example 6. *A rule that satisfies* efficiency*, the* endowment lower bound*,* drop strategy-proofness*,* upper invariance*, and* subset total drop strategy-proofness*, but not the strong endowment lower bound.*

Let $N = \{1, 2\}$, $O = \{a, b, c\}$, and $\omega = (ab, c)$. Let $\mathcal{Q} \subseteq \mathcal{R}^N$ be such that for each $R \in \mathcal{Q}$, $T(R_1, O) = c$ and $T(R_2, O) \neq c$. Let φ be defined by setting for each $R \in \mathcal{R}^N$,

$$\varphi(N, R, \omega) = \begin{cases} (c, ab) & \text{if } R \in \mathcal{Q} \\ G(N, R, \omega) & \text{if } R \in \mathcal{R}^N \setminus \mathcal{Q}. \end{cases}$$

It is trivial that φ satisfies *efficiency*, the *endowment lower bound*, and *upper invariance.* On the other hand, for each $R \in \mathcal{Q}$, while agent 1 owns two objects, they are assigned only one object. Hence, φ does not meet the *strong endowment lower bound.*

We show that φ is *drop strategy-proof.* Let $R \in \mathcal{R}^N$ and $i \in N$.

Case 1: $R \in \mathcal{Q}$.

Consider agent 1. If agent 1 drops either a or b to the bottom of their preferences, the resulting preference profile is in $\mathcal{Q}$. Hence, agent 1 receives the same assignment as in the case when they report R_1. If agent 1 drops c to the bottom of their preferences, the resulting preference profile is not in $\mathcal{Q}$. Then agent 1 is assigned ab. Because $c\ P_i\ a, b$, agent 1 has no incentive to operate such a misrepresentation.

Consider agent 2. Suppose that $R_2 : a, b, c$. If agent 2 drops either a or b to the bottom of their preferences, the resulting preference profile is $\mathcal{Q}$. Hence, agent 2 receives the same assignment as in the case when they report R_2. Suppose that $R_2 : a, c, b$. If agent 2 drops c to the bottom of their preferences, the resulting preference profile is in $\mathcal{Q}$. Hence, agent 2 receives the same assignment as in the case when they report R_2. If agent 2 drops a to the bottom of their preferences, resulting preference profile is not in $\mathcal{Q}$. Then agent 2 is assigned c. Because $a\ P_2\ c$, agent 2 has no incentive to operate such a misrepresentation.

Case 2: $R \notin \mathcal{Q}$. If suffices to consider situations where an agent misrepresents their preferences in such a way that the resulting preference profile is in $\mathcal{Q}$.

Consider agent 1. Suppose that $T(R_2, O) \neq c$. There are two possible preferences of agent 1 such that the resulting preference profile after agent 1 drops an object to the bottom of R_1 is in $\mathcal{Q}$, either $R_1 : a, c, b$ or $R_1 : b, c, a$. Because we apply an analogous argument for each of these cases, without loss of generality, we only show the first case.

If $R_2 : a, c, b$, then $\varphi(N, R, \omega) = (ab, c)$. If agent 1 drops a to the bottom of their preferences, agent 1 is assigned c. Because $a\ P_i\ c$, agent 1 has no incentive to operate such a misrepresentation. If $b\ R_2\ c$, then $\varphi(N, R, \omega) = (ac, b)$. If agent 1 drops a to the bottom of their preferences, agent 1 is assigned c. Because $a\ P_i\ c$, agent 1 has no incentive to operate such a misrepresentation.

Consider agent 2. Suppose that $T(R_1, O) = c$. Because $R \notin \mathcal{Q}$, $T(R_2, O) = c$. Then $\varphi(N, R, \omega) = \omega = (ab, c)$. Then, the only way of misrepresentation is that agent 2 drops c to the bottom of their preferences. However, if they do so, agent 2 is assigned ab. Because $c\ P_2\ a, b$, agent 2 has no incentive to operate such a misrepresentation.

Remark 3. The rule in Example 6 can be modified to be anonymous. We apply this variation to each two agent economy where one agent as two objects and the other agent has one object. We relabel them as above, and apply the rule.

E Proof of Proposition 2

Let $(N, R, \omega) \in \mathcal{E}$ be such that $N = \{1, 2\}$, $\omega = (a, bc)$, and

$$R_1 : b,\ c,\ a$$

$$R_2 : a,\ b,\ c.$$

Let φ satisfy *efficiency* and the *endowment lower bound*. Then

$$\varphi(N, R, \omega) \in \{(b, ac), (c, ab), (bc, a)\}.$$

$\underline{\varphi(N, R, \omega) = (b, ac)}$ Let $R_2' \in \mathcal{R}$ be such that

$$R_2' : b,\ a,\ c.$$

Note that this is a 2-drop, a push-up, or an adjacent swap for R_2. By *efficiency* and the *endowment lower bound*,

$$\varphi_2(N, R_1, R_2', \omega) = ab\ P_2\ ac = \varphi_2(N, R, \omega).$$

This violates each of *2-drop strategy-proofness*, *push-up strategy-proofness*, and *adjacent strategy-proofness*.

$\underline{\varphi(N, R, \omega) = (c, ab)}$ Let $R_1' \in \mathcal{R}$ be such that

$$R_1' : b,\ a,\ c.$$

Note that this is a drop, 2-push-up, or adjacent swap for R_1. By *efficiency* and the *endowment lower bound*,

$$\varphi_1(N, R_1', R_2, \omega) \in \{bc, b\}\ P_1\ c = \varphi_1(N, R, \omega).$$

This violates each of *drop strategy-proofness*, *2-push-up strategy-proofness*, and *adjacent strategy-proofness*.

$\underline{\varphi(N, R, \omega) = (bc, a)}$ Then

$$\varphi_2(N, R_1, R_2', \omega) = ab \; P_2 \; a = \varphi_2(N, R, \omega).$$

This violates each of *2-drop strategy-proofness*, *push-up strategy-proofness*, and *adjacent strategy-proofness*.

We have shown parts 2-4. For part 1, the only difference is the case of

$$\varphi(N, R, \omega) = (c, ab).$$

By *upper invariance*,

$$b \notin \varphi_1(N, R_1', R_2, \omega).$$

By the *endowment lower bound*,

$$\varphi(N, R_1', R_2, \omega) = (a, bc).$$

Since (b, ac) Pareto-dominates (a, bc) here, this contradicts the fact that φ is *efficient*.

We now embed the 2-agent examples above into economies with arbitrary number of agents. We construct preferences so that each of the additional agents ranks objects in their own endowment at the top, then show that each additional agent is assigned only their own endowment under φ. Analysis of the larger economy then reduces to the 2-agent example, and the desired result follows.

Let $E_1 = (\hat{N}, \hat{R}, \hat{\omega}) \in \mathcal{E}$ be such that $\hat{N} \supseteq \{1, 2\}$, $\hat{\omega}_1 = \omega_1$, $\hat{\omega}_2 = \omega_2$,

- $\hat{R}_1$ agree with R_1 when restricted to $\{a, b, c\}$,
- $\hat{R}_2$ agree with R_2 when restricted to $\{a, b, c\}$,
- $\hat{R}_1$ and $\hat{R}_2$ rank $\{a, b, c\}$ at the top, and
- for each $i \in \hat{N} \setminus \{1, 2\}$, each $x \in \hat{\omega}_i$, and each $y \notin \hat{\omega}_i$, $x \hat{P}_i y$.

Let $E_2 = (\hat{N}, \hat{R}_{N \setminus \{2\}}, \hat{R}_2', \hat{\omega}) \in \mathcal{E}$, where $\hat{R}_2'$ is such that $b \; \hat{P}_2' \; a \; \hat{P}_2' \; c$ and $\{a, b, c\}$ are at the top of singleton objects. Let $E_3 = (\hat{N}, \hat{R}_{N \setminus \{1\}}, \hat{R}_1', \hat{\omega}) \in \mathcal{E}$, where $\hat{R}_1'$ is such that $b\hat{P}_1' a \hat{P}_1' c$

and $\{a, b, c\}$ are at the top of singleton objects. By the *endowment lower bound*, for each $E \in \{E_1, E_2, E_3\}$, and each $i \in \hat{N} \setminus \{1, 2\}$, $\varphi_i(E) \supseteq \hat{\omega}_i$.

Consider E_3. We show that $\varphi_i(E_3) \subseteq \omega_i$. Suppose by contradiction that there is $i \in N \setminus \{1, 2\}$ such that $\varphi_i(E_3) \cap \{a, b, c\} \neq \emptyset$. If $a \in \varphi_i(E_3)$, then $\varphi_1(E_3) \cup \varphi_2(E_3) \subseteq \{b, c\}$, violating the *endowment lower bound* for either 1 or 2. If $b \in \varphi_i(E_3)$, then $\varphi_1(E_3) \cup \varphi_2(E_3) \subseteq \{a, c\}$, violating the *endowment lower bound* for either 1 or 2. If $c \in \varphi_i(E_3)$, $\varphi_1(E_3) \cup \varphi_2(E_3) \subseteq \{a, b\}$. By the *endowment lower bound*, $\varphi_1(E_3) = b$ and $\varphi_2(E_3) = a$. Let $\hat{R}''_2$ be a push-up for $\hat{R}_2$ by pushing up c.[24] By *efficiency* and the *endowment lower bound*,

$$\varphi_2(\hat{N}, \hat{R}_{N\setminus\{1,2\}}, \hat{R}'_1, \hat{R}''_2, \hat{\omega}) = ac\hat{P}_2 a = \varphi_2(E_3)$$

violating *push-up strategy-proofness*. Note that pushing up c is also a 2-drop for $\hat{R}_2$, so we can observe the same violation of *2-drop strategy-proofness*. Further, since $\hat{R}''_2$ is a 2-adjacent swap for $\hat{R}_2$, we can observe the same violation of *adjacent strategy-proofness*.[25]

Consider E_2. We show that $\varphi_i(E_2) \subseteq \omega_i$. Suppose by contradiction that there is $i \in N \setminus \{1, 2\}$ such that $\varphi_i(E_2) \cap \{a, b, c\} \neq \emptyset$. Let $x \in \{a, b, c\}$. If $x \in \varphi_i(E_2)$, then as above, $\varphi_1 \cup \varphi_2(E_2) \subseteq \{a, b, c\} \setminus \{x\}$, violating the *endowment lower bound* for either 1 or 2.

Consider E_1. We show that $\varphi_i(E_1) \subseteq \omega_i$. Suppose by contradiction that there is $i \in N \setminus \{1, 2\}$ such that $\varphi_i(E_1) \cap \{a, b, c\} \neq \emptyset$. If $a \in \varphi_i(E_1)$, then $\varphi_1(E_1) \cup \varphi_2(E_1) \subseteq \{b, c\}$, violating the *endowment lower bound* for either 1 or 2. If $b \in \varphi_i(E_1)$, then the *endowment lower bound*, $\varphi_1(E_1) = c$, and $\varphi_2(E_1) = a$. Similarly, if $c \in \varphi_i(E_1)$, then $\varphi_1(E_1) = b$ and $\varphi_2(E_1) = a$. Note that 2 receives a in the last two cases. Consider $\hat{R}'_2$ where 1 pushes up b. Above, we showed that at E_2, for each $i \in N \setminus \{1, 2\}$, $\varphi_i(E_2) = \omega_i$. Since $(\hat{N}, \hat{R}_{N\setminus\{2\}}, \hat{R}'_2, \hat{\omega}) = E_2$, by *efficiency* and the *endowment lower bound*, $\varphi_2(E_2) = ba\hat{P}_2 a = \varphi_2(E_1)$. This violates *push-up strategy-proofness* of φ. Note that $\hat{R}'_2$ is also a 2-drop and adjacent swap for $\hat{R}_2$, so we can make the same statement for *2-drop strategy-proofness* and *adjacent strategy-proofness*.

[24] We can replicate this step by considering this as a 2-drop (by a then b), or as two adjacent swaps (first c and b, then c and a).

[25] Let $\hat{R}'''_2$ rank $a\hat{R}'''_2 c\hat{R}'''_2 b$ at the top. Then $\hat{R}''_2$ is an adjacent swap for $\hat{R}'''_2$ (a and c), and $\hat{R}'''_2$ is an adjacent swap for $\hat{R}_2$ (b and c).

Thus for each $E \in \{E_1, E_2, E_3\}$ and each $i \in N\{1,2\}$, $\varphi_i(E) = \omega_i$, and the analysis of E reduces to the respective 2-agent problem.

Proof of Proposition 3.

Statement 1: Let R_j be a monotonic preference relation over bundles that is not lexicographic. We will construct an economy (N, R_j, R_{-j}, ω) such that $G(N, R, \omega)$ is not *efficient* at (N, R, ω). Let $N = \{i, j\}$; the resulting economy can be embedded into larger ones.

Let $R_i \in \mathcal{R}$ be a lexicographic preference relation such that R_i and R_j coincide when considering only singleton bundles (objects). Since R_j is not lexicographic, there exists $A, B \subseteq \mathbb{O}$ such that $A \; P_i \; B$ and $B \; P_j \; A$. Let $\omega = A \cup B$. Let $x \in \mathcal{X}(N, R, \omega)$ be such that $x = (A, B)$.

Case 1: Suppose that $A \cap B = \emptyset$. Let $\omega = (B, A)$. Since R_i and R_j coincide over singletons, $G_i(N, R, \omega) = B$ and $G_j(N, R, \omega) = A$. Then, x Pareto-dominates $G(N, R, \omega)$.

Case 2: Suppose that $A \cap B \neq \emptyset$. Notice that $A \setminus B \neq \emptyset$, otherwise $A \subsetneq B$, contradicting the lexicographic nature of R_i and $A \; P_i \; B$. Let $\omega = (B, A \setminus B)$. Since R_i and R_j coincide over singletons, $G(N, R, \omega) = (B, A \setminus B)$. Then, by $A \; P_j \; A \setminus B$, x Pareto-dominates $G(N, R, \omega)$.

Statement 2: Let R_0 be a preference relation over non-empty subsets of $\mathbb{O}$. Without loss of generality, let $a, b, c, d \in \mathbb{O}$ be such that

$$R_0 : a, b, c, d.$$

Let $\tilde{R}_0$ be a responsive preference relation over non-empty subsets of $\mathbb{O}$ such that

$$\tilde{R}_0 : a, b, c, d.$$

Let $N = \{1, 2\}$, $\omega_i \cup \omega_j = \{a, b, c, d\} \equiv O$, and $R_i = R_0$.

Suppose that for each R_j, where R_j is a responsive preference relation over non-empty subsets of $\mathbb{O}$, $G(N, R, \omega)$ meets the *endowment lower bound* at (N, R, ω). Let $A, B \subset O$ be such that $|A| = |B|$, $A \cap B \neq \emptyset$, and $A \; \tilde{P}_0 \; B$. So the definition of responsive preferences specifies the preference relation between A and B. Suppose that

$$B \; P_i \; A.$$

Let $\omega = (B, O \setminus B)$. Let R_j be such that

$$G(N, R, \omega) = (A, O \setminus A).$$

Note such a preference relation of agent j exists. Then $\omega_i \; P_i \; G_i(N, R, \omega)$. Hence, GTTC does not meet the *endowment lower bound* at (R, ω), a contradiction. $\square$