Trading Probabilities Along Cycles

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Abstract

Consider the problem of allocating indivisible objects when agents are endowed with fractional amounts and rules can assign lotteries. We study a natural generalization (to the probabilistic domain) of Gale's Top Trading Cycles. The latter features an algorithm wherein agents trade objects along a cycle—in our new family of rules, agents now trade *probabilities* of objects along a cycle. We ask if the attractive properties, namely *efficiency*, *individual rationality*, and *strategy-proofness* extended in the stochastic dominance sense, carry over to the Trading-Probabilities-Along-Cycles (TPAC) rules. All of these rules are *sd-efficient*. We characterize separately the subclass of TPAC rules satisfying the *sd-endowment lower bound* and *sd-strategy-proofness*. Regarding fairness, we follow in spirit to the *no-envy in net trade* condition of Schmeidler and Vind (1972), where the set of allocations satisfying the property essentially coincides with the set of competitive equilibria, and augment the notion appropriately for our environment. We further generalize the TPAC family while extending results on *sd-efficiency* and the *sd-endowment lower bound*, and provide sufficient conditions on parameters for the rules to arbitrarily closely satisfy the *sd-no-envy in net trade*.

Keywords Top trading cycles; probabilistic assignment; efficiency; strategy-proofness; individual rationality; no-envy in net trade

JEL Classification C78; D61; D63; D70

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1 Introduction

A city public housing authority wishes to reshuffle tenants due to demographic changes. For example, family sizes fluctuate and there could be cases of one person living in a large apartment while a family of five resides in a small space. In the reshuffling process, the authority may have various goals including: As much as possible, place tenants where they want to live while enacting some type of fairness if multiple people apply for the same apartment. Certain tenants should have sufficient chance to be in certain arrangements e.g. senior tenants should have sufficient chance to be close to particular resources, or persons with disabilities have first priority at buildings with accommodations.

New York City provides various forms of public housing for over half a million residents and recently faced this complicated task.[1] It is imaginable that many other cities face similar problems. How should they design such processes?

We model this problem as one of re-allocating objects (apartments) to agents wherein agents have preferences over objects, and probabilistic assignment—defined by a lottery over objects—is possible. To express various guarantees that the authority may wish to respect, we consider the environment where agents also have rights to certain objects. These rights are indicated by fractional ownership of the objects and represent a lower bound on their welfare. We study a large family of probabilistic rules that contain as a special case Gale's Top Trading Cycles (TTC), and view them as a natural generalization (Shapley and Scarf, 1974). Our results shed light on tradeoffs regarding efficiency, manipulability, and fairness when we move to the probabilistic domain.

In deterministic environments, whether objects are initially owned or have attached priorities, TTC is ubiquitous. When each agent owns one and consumes one object, it is the only *efficient*, *individually rational*, and *strategy-proof* rule (Ma, 1994; Sönmez, 1999; Anno, 2015). Dropping *individual rationality*, the expanded set of rules may be defined as an outcome of an algorithm where agents trade along cycles given a particular ownership structure (Pápai, 2000; Pycia and Ünver, 2017). In school choice where students have affixed *priorities* as opposed to ownership, TTC can still be used to define an *efficient* and *strategy-proof* mechanism that satisfies further fairness criteria (Abdulkadiroğlu and Sönmez, 2003; Dur, 2013; Dur and Morrill, 2017; Morrill, 2013, 2015a,b).

We ask the following question: Taking the intuition of trading objects along cycles as a starting point, does the procedure maintain its attractiveness in the more general environment of probabilistic assignment? That is, instead of simply trading entire objects, does trading *probabilities* result in satisfactory rules? If i owns 0.2 of a and j owns 0.4 of b, then

[1] Harris, Elizabeth. "Alone in Public Housing, With a Spare Bedroom." New York Times March 11, 2012.

it seems entirely reasonable for them to trade their shares if they wish.

To answer, we define a large family of probabilistic rules built upon such a premise. We start with the Trading-Probabilities-Along-Cycles (TPAC) family, expand to the Generalized TPAC family, and along the way study their axiomatic properties. Each rule in the TPAC family is defined by a TTC-like algorithm wherein agents have rights to trade or consume fractions of objects. Since several agents may have the right to trade fractions of the same object, each rule also specifies a particular order in which the trading rights are respected. Different trading rights and priority orders define different rules. Thus, we can define a family of TPAC rules, each associated with two exogenously given parameters encoding trading rights and priority orders.

Since agents may be assigned lotteries over objects, we extend their preferences over objects to preferences over lotteries by means of stochastic dominance, and consider variants of *efficiency*, *strategy-proofness*, and *individual rationality* appropriate to the probabilistic environment (Bogomolnaia and Moulin, 2001).[2] We indicate these variants with the "*sd-*" preface.

Our first contribution is to show to what extent this probabilistic extension of TTC satisfies the probabilistic analog of the original three properties characterizing it. The first two results are encouraging and straightforward: Each rule in TPAC family satisfies *sd-efficiency* (Proposition 1). For each fractional endowment profile, we characterize the subfamily of TPAC satisfying the *sd-endowment lower bound* (Proposition 2). Our next result illustrates a known trade-off between the three properties in rich environments: If there is even a single agent with positive fractional endowment of more than one object, then there is no rule in the TPAC family that satisfies the *sd-endowment lower bound* and *sd-strategy-proofness* (Theorem 1). The intuition and success of trading along cycles runs into serious difficulties when we allow for the least bit of fractional endowment. A direct corollary of this result is that, within the TPAC family, TTC is the only *sd-strategy-proof* rule (Corollary 1).

Our second contribution is to show that interesting notions of *no-envy* are achievable. For the classical exchange problem, Schmeidler and Vind (1972) proposes a *no-envy* concept that accounts for agents' possibly unequal endowments. At a proposed allocation, each agent moves from their endowment to their assignment—resulting in a net positive trade of some objects and negative of others; their *no-envy in net trade* property requires that each agent prefers their own net trade to any other's. They show that the concept is tightly

[2]We use the *ordinal efficiency* notion introduced by Bogomolnaia and Moulin (2001). McLennan (2002) shows that any *ordinally efficient* allocation maximizes the sum of expected utilities for some profile of vNM utility indices consistent with the original ordinal preference profile.

linked with the set of competitive equilibrium allocations.[3] We adopt this fairness notion for probabilistic assignment with endowments. Then, we introduce the Generalized TPAC family of rules wherein each member operates by allowing agents to appear *multiple times* within the order of trading rights; this allows agents to alternate in the order of trading rights. We show that we can achieve with arbitrary closeness *sd-no-envy in net trade*, and provide sufficient conditions on the parameters to do so (Theorem 2). The *sd-efficiency* and *sd-endowment lower bound* properties of TPAC family are also preserved by the Generalized TPAC family (Propositions 3 and 4).

Our result hints at known connection between TTC algorithms and competitive equilibria. In the original housing market, the allocation obtained by TTC can be supported as a competitive equilibrium (Shapley and Scarf, 1974). In school choice where agents have priorities instead of endowments, a type of competitive equilibrium is unique and coincides with the (school choice) TTC allocation (Dur and Morrill, 2017). Although we do not define a competitive notion here, our TTC-inspired mechanisms can satisfy one of its fundamental *no-envy* properties, namely *no-envy in net trade*.

We discuss our results in relation to the probabilistic assignment literature. Hylland and Zeckhauser (1979) introduces the problem and imagines a pseudo-market mechanism wherein each agent is endowed with an income and purchases probabilities of objects according to prices. Note that in contrast to our model, agents do not own fractions of objects as a primitive. Their mechanism is *ex-ante efficient*, but is not *strategy-proof*. While they employ the intuition of the competitive equilibrium, we conduct a parallel exercise except with TTC. Unfortunately, there is no rule satisfying *sd-efficiency*, *sd-endowment lower bound*, and *sd-strategy-proofness* when considering an equal division of the endowments (Athanassoglou and Sethuraman, 2011). Weaker notions of *sd-strategy-proofness* (requiring only that any outcome from a lie does not stochastically dominate the truth) does not recover compatibility (Aziz, 2018). Both of the previous papers prove their results at particular endowment profiles e.g. equal division in the former. Within the TPAC family, we confirm that some special characteristic of the previous endowment profiles, e.g. full support or a type of cycle in the support, is not driving the incompatibility; rather, it is pervasive, holding for *all* non-extreme points (Theorem 1). In light of these impossibilities, the focus turned to fairness and several papers proposed rules satisfying notions based on equal treatment or elimination of justified envy (Athanassoglou and Sethuraman, 2011; Echenique et al., 2021; Kesten, 2009; Yılmaz, 2010).

Closest to ours are Aziz (2015) and Yu and Zhang (2021). Additionally allowing indif-

[3] Each competitive equilibrium allocation satisfies *no-envy in net trade*, and each *strong no-envy in net trade* allocation can be supported as a competitive equilibrium.

ference in preferences, Aziz (2015) defines a class of rules satisfying *sd-efficiency* and the *sd-endowment lower bound*. Yu and Zhang (2021) define a large class of rules relying on an algorithm that is procedurally significantly different from ours. The key difference is that in each step of their algorithm, each object points to each agent that owns positive amount of it. Subsequently, when each agent points to their most preferred object, the resulting graph can be complex with multiple overlapping cycles. They encode this information into system of linear equations and constraints representing feasibility. Within the constraints, there are additional exogenous parameters that specify for each object and each owner of the object the "share" that they *must* use of the object to trade. The assignment at this step is the maximum solution to the system with constraints. Contrast this with our algorithm, which is a rather straightforward extension of Gale's Top Trading Cycles with trading rights and orders. When restricted to strict preferences, Yu and Zhang (2021) rules are a strict superset of our TPAC family which in turn is a strict superset of those in Aziz (2015).

We are also related to papers that employ *mixtures* of TTC rules to recapture fairness. Interestingly, Random Serial Priority is equivalent to the Core from Random Endowments (Abdulkadiroğlu and Sönmez, 1998; Knuth, 1996).[4] Although equal treatment and no-envy were the focus of these papers, such mixtures can satisfy the *sd-endowment lower bound* for some fractional endowment profile by varying the weights placed on each component rule. For example, if an agent owns 10% of object a and 20% of object b, then the mixture can place 10% weight on a rule where the agent initially owns a and 20% weight on a rule where the agent owns b. Harless and Phan (2019) perform this type of analysis, but for the environment where each agent partially owns only one object.[5] These mixtures are *sd-strategy-proof* but not generally *sd-efficient*; in contrast, our TPAC rules are *sd-efficient* but not generally *sd-strategy-proof*.

The paper is organized as follows. In Section 2, we define the probabilistic assignment problem. In Section 3, we define several desirable properties of an allocation rule. In Section 4, we define the TPAC family of rules, and we state properties of this family in Section 5. Pivoting to fairness, we discuss *no-envy in net trade* and the Generalized TPAC family in Section 6. We conclude in the final section.

[4]This result has been extended to more general families of TTC rules (Bade, 2020; Bogomolnaia and Moulin, 2001; Carroll, 2014; Che and Kojima, 2010; Lee and Sethuraman, 2011).

[5]We consider a more general endowment structure than Harless and Phan (2019). In our model, an agent can own fractions of several different objects; in Harless and Phan (2019), each agent only owns share of one object, and no two agents own share of the same object. Also, in their paper, they consider rules that are convex combinations of discrete *efficient* and *group strategy-proof* rules (those in Pápai (2000) and Pycia and Ünver (2017)).

2 Model

Let $O \equiv \{a, b, c, \dots\}$ be a finite set of objects, and $N \equiv \{1, 2, 3, \dots\}$ be a finite set of agents with $|O| = |N| \geq 3$. Each agent $i \in N$ has a strict **preference relation** R_i over objects. Let $\mathcal{R}$ be the set of all strict preference relations, and $R \equiv (R_i)_{i \in N} \in \mathcal{R}^N$ be a **preference profile**.

The first difference from the standard object allocation problem with discrete endowment profiles is that for each object, there may be a set of agents *each* with some guarantee to it. To capture this idea, each agent is allowed to own a fractional amount of each object. Let $\triangle O \equiv \{(x_o)_{o \in O} \in [0,1]^O : \sum_{o \in O} x_o = 1\}$ be the set of possible fractional endowments defined on O. Each agent $i \in N$ has an **endowment** $\omega_i \equiv (\omega_{io})_{o \in O} \in \triangle O$. Let $Z \equiv \{(x_i)_{i \in N} \in (\triangle O)^N : \forall o \in O, \sum_{i \in N} x_{io} = 1\}$ be the set of all lists of jointly feasible endowments. An **endowment profile** $\omega \equiv (\omega_i)_{i \in N} \in Z$ specifies an endowment for each agent wherein the total amount owned for each single object is 1.

Suppose that there are two agents that have the same preferences over the set of objects. Either one of the agents will be favored against the other one at any deterministic allocation. That is, fairness becomes a fundamental issue for deterministic allocation rules. In order to recover some fairness, we instead distribute the objects via lotteries. For each $i \in N$ and each $o \in O$, let $x_{io} \in [0,1]$ be the probability of agent i receiving object o. An **assignment** for agent i is a probability distribution (or lottery) over O, $x_i \equiv (x_{io})_{o \in O} \in \triangle O$. An **allocation** is a list $x \equiv (x_i)_{i \in N} \in Z$.[6] A **rule** $\varphi : \mathcal{R}^N \to Z$ recommends an allocation for each preference profile.

We extend the preference relation of each agent, R_i, to preferences over lotteries by means of first-order stochastic dominance: Let $i \in N$, $x_i, x'_i \in \triangle O$, and $R_i \in \mathcal{R}$. Then, agent i finds x_i at least as good as x'_i, $x_i \; R_i^{sd} \; x'_i$, if for each $o \in O$,

$$\sum_{a \in O : a R_i o} x_{ia} \geq \sum_{a \in O : a R_i o} x'_{ia}.$$

If there is at least one strict inequality, then $\boldsymbol{x_i}$ **stochastically dominates** $\boldsymbol{x'_i}$ **at** $\boldsymbol{R_i}$ and we write $x_i \; P_i^{sd} \; x'_i$. Thus, if $x_i \neq x'_i$ and $x_i \; R_i^{sd} \; x'_i$, we have $x_i \; P_i^{sd} \; x'_i$. Equivalently, an agent prefers a lottery to another according to the first-order-stochastic-dominance extension if the following holds: the former yields higher expected utility than the latter with respect to any von Neumann Morgenstern utility function compatible with their preference relation over objects. For each pair of allocations $x, x' \in Z$, $\boldsymbol{x}$ **stochastically dominates** $\boldsymbol{x'}$ **at** $\boldsymbol{R}$, if for

[6] By the Birkhoff-von Neumann Theorem, x can be expressed as a convex combination of permutation matrices (Birkhoff, 1946; von Neumann, 1953).

each $i \in N$, $x_i \; R_i^{sd} \; x'_i$, and there is $j \in N$ such that $x_j \; P_j^{sd} \; x'_j$.

3 Axioms

The first requirement is that the rule does not waste resources.

Sd-efficiency: (Bogomolnaia and Moulin, 2001) For each $R \in \mathcal{R}^N$, there is no $x \in Z$ such that x stochastically dominates $\varphi(R)$ at R.

The second requirement is that the rule respects ownership. The most natural way is to make each agent at least as well off as they would be by consuming their own endowment.

Sd-endowment lower bound: For each $R \in \mathcal{R}^N$, and each $i \in N$,

$$\varphi_i(R) \; R_i^{sd} \; \omega_i.$$

The last is the ubiquitous incentive compatibility requirement in the design of allocation rules: the assignment that an agent receives when they tell the truth is at least as good as the assignment that they receive when they lie.

Sd-strategy-proofness: For each $i \in N$, each $R \in \mathcal{R}^N$, and each $R'_i \in \mathcal{R}$,

$$\varphi_i(R) \; R_i^{sd} \; \varphi_i(R'_i, R_{-i}).$$

In Section 6 we also consider fairness properties, and we delay discussion until that section.

Remark 1. (Athanassoglou and Sethuraman, 2011) At the equal endowment profile, there is no rule that satisfies *sd-efficiency*, *sd-endowment lower bound*, and *sd-strategy-proofness*.

They show that impossibility holds at the equal endowment profile. Aziz (2018) proves a similar incompatibility for a weakening of *sd-strategy-proofness*, also at a particular endowment profile. We explore general endowment structures in the following section.

4 Trading-Probabilities-Along-Cycles Family of Rules

In this section, we define a large class of probabilistic rules that extends the intuition underlying TTC. Each rule is parametrized by 1) the amount of each object that each agent has the right to trade, and 2) for each object, the order in which agents have the right to trade their share of the object. Once this ownership structure is defined, the usual TTC step can be used. As the algorithm proceeds, the current owner of each object updates according to the priority parameter as the right to trade expires.

More formally, for each $i \in N$, and each $o \in O$, let $r_{io} \in [0, 1]$ be the **trading right of agent $\boldsymbol{i}$ for object $\boldsymbol{o}$**, and $r_i \equiv (r_{io})_{o \in O} \in \triangle O$ be the **trading right of agent $\boldsymbol{i}$**. Let $r \equiv (r_i)_{i \in N} \in Z$ be a **trading rights profile**. For each $o \in O$, let $\sigma_o : N \to \{1, \ldots, |N|\}$ be a bijection and the **priority order for object $\boldsymbol{o}$**. For each $i \in N$, $\sigma_o(i)$ defines the position of agent i in the priority order for o. Let $\sigma \equiv (\sigma_o)_{o \in O}$ be an **object priority profile**.

We refer to each pair (r, σ) as a trading-probabilities-along-cycles (TPAC) parameter. For each pair (r, σ), let $\varphi^{(r,\sigma)}$ be the associated **Trading-Probabilities-Along-Cycles** rule.[7] Finally, for each preference profile $R \in \mathcal{R}^N$, we use the following algorithm to compute $\varphi^{(r,\sigma)}(R)$:

> **Step 1**: Construct a weighted, directed graph as follows: The set of vertices is the set of agents and objects. For each agent i, there is a directed edge with weight 1 to their most preferred object according to R_i. For each object o, let $j = \sigma_o^{-1}(1)$ be the highest priority agent for o, and let there be a directed edge with weight r_{jo} from o to j. At least one cycle exists. For each cycle, there is an edge with minimum weight w among edges in the cycle. For each agent i in the cycle, 1) assign i this amount w of the object o for which there is an edge from i to o, and 2) decrease by this amount w i's trading right of the object o' in the cycle for which there is an edge from o' to i .
>
> For each $s \geq 1$, let r^s be the updated trading rights profile at the end of Step s.
>
> **Step $\boldsymbol{s}$**: Construct a weighted, directed graph as follows: The set of vertices is the set of 1) agents who have not been assigned a total amount of 1 of objects, and 2) objects for which there is $i \in N$ with $r_{io}^{s-1} > 0$. For each agent i in the graph, there is a directed edge with weight $\sum_{o \in O} r_{io}^{s-1}$ from i to their most preferred object in the graph according to R_i. For each object o in the graph, let j be the highest priority agent with positive trading right of o, that is, $r_{jo}^{s-1} > 0$ and for each other j' in the graph

[7] We distinguish between the endowment profile ω and the trading rights parameter r used by the rule for several reasons. First, the family can be defined independent of endowment information and thus used in situations without the latter. Second, when we generalize upon them in Section 6, we will also define rules operationalized on two parameters that similarly reflect the concepts of trading rights and an associated order. To be consistent across the two families and highlight the parallel intuition, we explicitly define both parameters in both families. Finally, the trading rights in the generalized family is a different mathematical object than an endowment profile.

with $r^{s-1}_{j'o} > 0$, $\sigma_o(j) < \sigma_o(j')$. Let there be a directed edge with weight r^{s-1}_{jo} from o to j. At least one cycle exists. For each cycle, there is an edge with minimum weight w among edges in the cycle. For each agent i in the cycle, 1) assign i this amount w of the object o for which there is an edge from i to o, and 2) decrease by this amount w i's trading right of the object o' in the cycle for which there is an edge from o' to i .

Since there are finite numbers of agents and objects, the algorithm ends in finitely many steps. In the end, each agent is assigned a lottery and each object is exhausted. □

The TPAC family (and its subsequent generalization) can be seen as a way to select among possibly overlapping trading cycles when each object may have multiple, partial owners and points to each of them. The priority order and trading rights of objects determine the order and extent to which cycles are chosen.

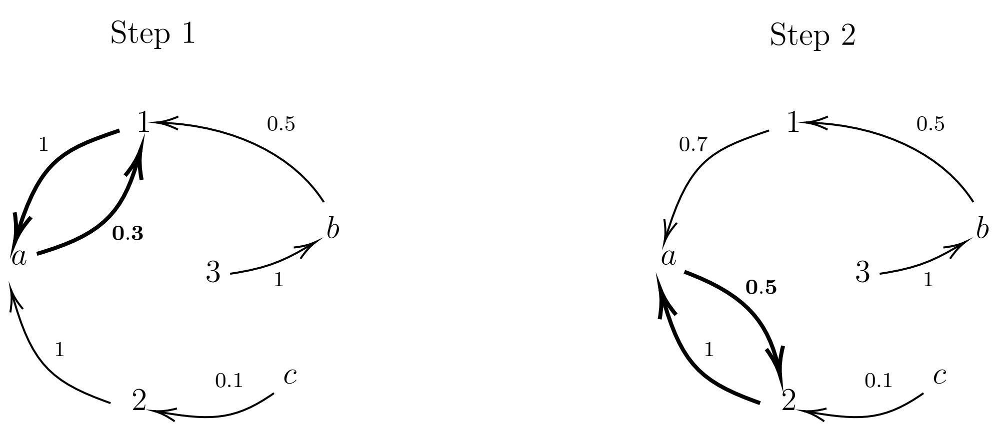
Next, we illustrate the algorithm.

Example 1. Let $O = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Let $R \in \mathcal{R}^N$ and (r, σ) be a TPAC parameter as follows. We apply the algorithm to compute φ.

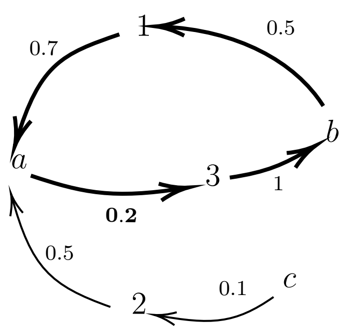
R_1	R_2	R_3
a	a	b
b	c	a
c	b	c

σ_a	σ_b	σ_c
1	1	2
2	3	3
3	2	1

r	a	b	c
1	0.3	0.5	0.2
2	0.5	0.4	0.1
3	0.2	0.1	0.7



Step 3



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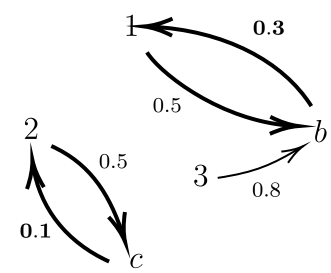

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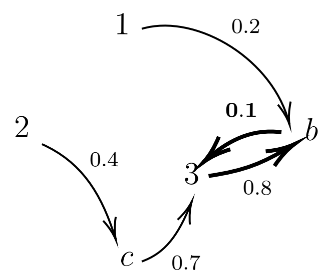
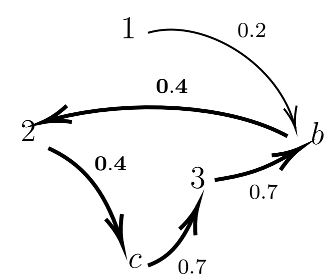
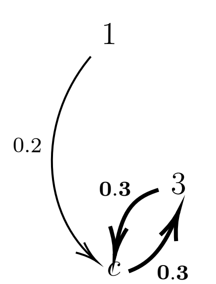


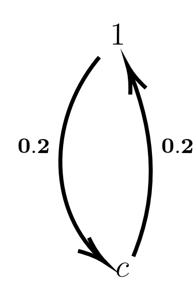
Step 6



Step 7



Step 8



In Step 1, the only cycle is formed by $\{a, 1\}$ with minimum weight equal to 0.3. Agent 1 receives 0.3 of a; their trading right of a decreases by 0.3 and is updated to 0. In Step 2, the only cycle is formed by $\{a, 2\}$ with minimum weight equal to 0.5. Agent 2 receives 0.5 of object a; their trading right of a decreases by 0.5 and is updated to 0. In Step 3, the only cycle is formed by $\{a, 3, b, 1\}$ with minimum weight equal to 0.2. Agent 1 receives 0.2 of a; their trading right of b decreases by 0.2 and is updated to 0.3. Agent 3 receives 0.2 of b; their trading right of a decreases by 0.2 and is updated to 0. Object a is fully exhausted and removed from the graph. The algorithm proceeds similarly and terminates at Step 8. The allocation selected by $\varphi^{(r,\sigma)}$ at the preference profile R is as follows:

$\varphi^{(r,\sigma)}(R)$	a	b	c
1	0.5	0.3	0.2
2	0.5	0	0.5
3	0	0.7	0.3

Remark 2. Suppose that each agent owns exactly one object. That is, $\omega \in Z$ is such that for each $i \in N$ and each $o \in O$, $\omega_{io} \in \{0,1\}$. Now, for each TPAC parameter (r,σ) with $r = \omega$, $\varphi^{(r,\sigma)}$ coincides with TTC using ω as the endowment profile.

5 Properties of the TPAC Family

Encouragingly, each TPAC rule satisfies several desirable properties.

First, we show that for each preference profile, no allocation stochastically dominates the allocation chosen by a TPAC rule.

Proposition 1. *Each TPAC rule satisfies* sd-efficiency.

Proof. Let (r,σ) be a TPAC parameter. Suppose by contradiction that there is $R \in \mathcal{R}^N$ such that there is an allocation that stochastically dominates $\varphi^{(r,\sigma)}$ at R. Let $x \equiv \varphi^{(r,\sigma)}(R)$. By Lemma 3 of Bogomolnaia and Moulin (2001), there is a sequence of agents (relabelling if necessary) $1,\ldots,k$ and objects $o_1,\ldots,o_k$ such that for each $i \in \{1,\ldots,\bar{k}\}$, $x_{io_{i+1}} > 0$ and $o_i \; R_i \; o_{i+1} \mod \bar{k}$.

Consider agent 1. Let s_1 be the first step in the algorithm for $\varphi^{(r,\sigma)}$ that 1 forms a directed edge to o_2. There is such a step as $x_{1o_2} > 0$. This implies that each object $o \in O$ with $o \; P_1 \; o_2$ is no longer available (mass 1 of o has been assigned to various agents); in particular, o_1 is no longer available. Furthermore, this implies that at Step s_1, agent 2 forms a directed edge at o_2 or an object preferred to o_2 with respect to R_2.

Let s_2 be the first step in the algorithm that 2 forms a directed edge to o_3. By construction of R_2, this implies that o_2 is no longer available, and subsequently $s_2 > s_1$.

In general, for each $k \in \{1,\ldots\bar{k}\}$, let Step s_k be the first step that k forms a directed edge to o_{k+1}. Following the reasoning above, for each $k \in \{1,\ldots\bar{k}\}$, $s_k > s_{k-1}$.

At Step $s_{\bar{k}}$, there is a directed edge from $\bar{k}$ to o_1. This contradicts the fact that o_1 is no longer available. ∎

Next, we show that a TPAC rule assigns each agent a lottery that they find at least as desirable as their endowment if and only if the amount of each object that an agent has the right to trade is at least as large as the amount of each object that they own.

Proposition 2. *The TPAC rule $\varphi^{(r,\sigma)}$ satisfies the* sd-endowment lower bound *if and only if* $r = \omega$.

Proof. Let $R \in \mathcal{R}^N$, (r,σ) be a TPAC parameter, and $x \equiv \varphi^{(r,\sigma)}(R)$.

Let (r,σ) be such that $r = \omega$, and suppose that for agent $i \in N$, $R_i = o_1 o_2 \dots o_k$ where $k = |O|$.[8] By definition of a TPAC rule, we have

$$\begin{aligned}
x_{io_1} &\geq r_{io_1} \\
x_{io_1} + x_{io_2} &\geq r_{io_1} + r_{io_2} \\
\vdots \qquad & \vdots \quad \vdots \\
x_{io_1} + x_{io_2} + \ldots + x_{io_k} &= r_{io_1} + r_{io_2} + \ldots + r_{io_k} = 1.
\end{aligned}$$

The first inequality follows from the fact that agent i consumes their own trading right of object o_1 as well as an additional amount from trading away other objects. The subsequent inequalities are similar. Hence, $\varphi^{(r,\sigma)}$ satisfies the *sd-endowment lower bound*.

Now, let $\varphi^{(r,\sigma)}$ satisfy the *sd-endowment lower bound*. Suppose by contradiction that $r \neq \omega$. Then there is an agent $i \in N$ and an object $o \in O$ such that $r_{io} < \omega_{io}$. Let $R \in \mathcal{R}^N$ be defined as follows: For each $i \in N$ and each $o' \in O$, $o \; R_i \; o'$. By definition of a TPAC rule, for each $i \in N$, $x_{io} = r_{io}$. Since $x_{io} < \omega_{io}$, x_i does not stochastically dominate ω_i at preference relation R_i, in contradiction to $\varphi^{(r,\sigma)}$ satisfies the *sd-endowment lower bound*. $\square$

Next, we check the compatibility of the *sd-endowment lower bound* and *sd-strategy-proofness.*

Theorem 1. *Suppose that there is an agent with positive fractional endowment of more than one object. Then, no TPAC rule satisfies the* sd-endowment lower bound *and* sd-strategy-proofness.

The proof of Theorem 1 is provided in Appendix A. We first establish several lemmas showing that certain classes of TPAC parameters are not *sd-strategy-proof.* For example, if (r,σ) is such that there are two agents i and j with first priority for objects a and b, and both have positive probability of another object c, then $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.* We then generalize this to all possible parameters by directly applying the lemmas, or by constructing an infinite sequence of agents—contradicting finiteness of N.

[8] The preferences of agent i that prefers o_1 to o_2, o_2 to o_3 and so on is written as $R_i = o_1 o_2 \dots o_k$.

Thus, a TPAC rule wherein the trading rights parameter r is in the simplex is *sd-strategy-proof* if and only if each agent owns one object.

Corollary 1. *TTC is the only* sd-strategy-proof *TPAC rule.*

With regards to incentive compatibility, we are able to give a definitive but disappointing answer to the question of whether or not trading along cycles gives rise to satisfactory rules.

6 Fairness and TPAC

In this section we turn our focus to fairness amongst the agents. We adopt the notion of Schmeidler and Vind (1972) wherein the *no-envy* concept of Foley (1967) is defined for the classical exchange problem to accommodate inequalities in agents' initial endowments. We follow in spirit and study a version of it in the probabilistic assignment problem with endowments. Let an agent's *net trade* be the vector defined by their assignment less their endowment. The condition of Schmeidler and Vind (1972) requires that agent i should not prefer agent j's net trade over their own.

In both classical exchange and our environment, however, adding another agent's net trade to one's endowment may result in a point outside of their consumption space. For example, if i has 0.2 endowment of a, and j's net trade of a is -0.4, then applying j's net trade to i's endowment gives i -0.2 of a—the property would not be well-defined. Schmeidler and Vind (1972) take the approach of altogether dropping the *no-envy* requirement in this scenario; thus, they invoke *no-envy* when net trades are feasible, and otherwise not.

Our notion takes a more progressive approach. In the above scenario, note that if we scale back the net trade of j by $\frac{1}{2}$, then applying it to i's endowment can result in an assignment. This fraction $\frac{1}{2}$ can be seen as a normalization that accounts for the inequalities in agents' initial endowments, and, further, guarantees comparability. More generally, the closer i and j's endowment is, the closer the normalization is to 1, and so *no-envy* is applied with greater strength. We thus capture the spirit of the Schmeidler and Vind (1972) condition: an agent never prefers any other agent's normalized net trade to their own. In summary, we conceptually strengthen parts of their property while weakening others. The key intuition is that their binary approach fully compares some but ignores the remaining net trades, while we take all net trades into account by monotonic adjustment.

In this section, our main result is to show that with a generalization of the TPAC family, it is possible to achieve arbitrarily closely this property, and we give sufficient conditions to do so.

6.1 No-Envy in Net Trade

For each agent $i \in N$ and each assignment $x_i \in \triangle O$, let ***i*'s net trade at $\boldsymbol{x_i}$** be $t_i(x_i) = x_i - \omega_i$. When the allocation is clear, we denote $t_i(x_i)$ by t_i. For each $i, j \in N$, and each allocation $x \in Z$, let ***i*'s *j*-net trade at $\boldsymbol{x}$** be $x_i^j \in \triangle O$ such that $x_i^j = \omega_i + \alpha(\omega_i, \omega_j)t_j$ where $\alpha(\omega_i, \omega_j) = \min_{o \in O: \omega_{jo} > 0}\{1, \frac{\omega_{io}}{\omega_{jo}}\}$.[9] As mentioned, the adjustment of $\alpha(\omega_i, \omega_j)$ of j's trade vector ensures that i's j-net trade at x is within their consumption space. For brevity, we denote $\alpha(\omega_i, \omega_j)$ by $\alpha(i, j)$.

Sd-no-envy in net trade: For each $R \in \mathcal{R}^N$, and each $i, j \in N$,

$$\varphi_i(R)\, R_i^{sd}\, \varphi_i^j(R).$$

When two agents have the same endowment, this property is the *sd-no-envy* condition. If, in addition, the two agents have the same preference over objects, then the property requires that the two agents receive the same assignment.

We define the approximate version of *sd-no-envy in net trade*. Let $\epsilon \geq 0$ be the relaxation on each inequality constraint of the property.

$\boldsymbol{\epsilon}$-Sd-no-envy in net trade: For each $R \in \mathcal{R}^N$, each $i, j \in N$, and each $o \in O$,

$$\epsilon + \sum_{b:\, b\, R_i\, o} \varphi_{ib}(R) \geq \sum_{b:\, b\, R_i\, o} \varphi_{ib}^j(R).$$

We make several remarks about the property. First, it embodies a natural "monotonicity of comparison": If two agents have the same traits, then we should treat them similarly; if they are very different, then it is unclear on how to compare their outcomes. If two agents i and j have the same endowment, then $\alpha(i, j) = 1$ and the property *directly* compares their net trades. As the distance between their endowments increase, $\alpha(i, j)$ decreases and the property is more agnostic about comparison of their net trades. Second, it leaves room for further notions of fairness. For example, suppose that both i and j both top-rank a "basic necessity" object a, but i is endowed with much more of a. Both i and j would like to trade the same amount of b for c. The designer favors j in this aspect, because of j's disadvantage of having less a. Lastly, from an informational standpoint, α is parsimonious—it depends on only ω and can be considered independent of the rule, allocation, or preferences at hand.

Consider alternatively allowing the net trade normalization to vary across ω *and allocations*. This may require, for example, agents with different endowments to nevertheless

[9] We show that x_i^j is indeed in $\triangle O$ in Appendix B.

directly compare net trades whenever feasible. Let this normalization be called $\hat{\alpha}$.[10] Consider the three points in the previous paragraph. The monotonicity of comparison is no longer present and may, in fact, be reversed: at some allocations, agents i and j with similar endowments have a weak comparison of net trades ($\hat{\alpha}(\omega_i, \omega_j, x)$ is low), while agents i and k with different endowments have a direct comparison ($\hat{\alpha}(\omega_i, \omega_k, x)$ is high). The notion of fairness adjusting for basic necessities may not be possible. For example, the alternative property may require that i and j trade exactly the same amounts of b and c, because that net trade is feasible for both. Lastly, $\hat{\alpha}$ varies substantially across allocations.

We mention that Yu and Zhang (2021) consider an interesting and related but logically independent notion. They directly relax the *no-envy* stochastic dominance constraints, allowing for more violation as two agents' endowments increasingly differ. Their *bounded envy* requires that for each pair of agents i and j, envy is bounded by the aggregate difference between their endowments for objects that i has less of than j.[11] Conceptually, they compare final allocations, but adjust the *no-envy* constraint; in the reverse order, we adjust net trades, then compare the resulting hypothetical assignments.

6.2 The Generalized TPAC Family

In this section, we introduce a natural generalization of the TPAC family. Recall that in the TPAC rule associated with (r, σ), the parameter σ is a priority order of the agents indicating when an agent has the right to trade an object. Notice that each agent appears exactly once in this order. We now allow for agents to appear multiple times and specifying, for each time an agent appears, the amount of trading right they have. For example, if both i and j each own 0.3 of a, then TPAC requires that either i or j exercise full trading rights of a first, while now, i and j can alternate three times exercising 0.1 trading right of a. The mechanism can thus procedurally *favor* a particular agent by setting more of their trading rights first, or *equalize* across agents by repeatedly alternating amongst them smaller trading rights.[12]

[10] Formally, for each $i, j \in N$, each endowment $\omega \in Z$, and each allocation $x \in Z$, let $\hat{\alpha}(\omega_i, \omega_j, x) = \max_{\beta \in [0,1]:\ \omega_i + \beta t_j(x_j) \in \triangle O} \beta$. Then, use $\hat{\alpha}$ in place of α in the definition of *sd-no-envy in net trade*.

[11] The following example demonstrates the two properties' independence. Let $O = \{a, b, c, d, e\}$, $\omega_i = (0.1, 0.2, 0.2, 0.5, 0)$, and $\omega_j = (0.4, 0.4, 0.2, 0, 0)$, $R_i = R_j :\ e, a, b, c, d$. Note that $\alpha(i, j) = \frac{1}{4}$, and the bound of Yu and Zhang (2020) is 0.5. The assignment where $x_i = (0.1, 0, 0.2, 0.5, 0.2)$ and $x_j = (0.1, 0, 0.2, 0, 0.7)$ satisfy both properties when restricted to i and j. The assignment where $x_i = (0.1, 0.2, 0.2, 0.5, 0)$ and $x_j = (0, 0.4, 0.2, 0, 0.4)$ satisfies their *bounded envy* restricted to i and j, but not our *sd-no-envy in net trade*. The assignment where $x_i = (0.1, 0, 0.2, 0.5, 0.2)$ and $x_j = (0, 0, 0.2, 0, 0.8)$ satisfies *sd-no-envy in net trade* restricted to i and j, but not *bounded envy*.

[12] The spirit of our mechanism is reminiscent to that of the family of Probabilistic Serial rules of Bogomolnaia and Moulin (2001) wherein "eating speeds" of the agents can be varied. Their environment, though, is

More formally, for each $o \in O$, let $\bar{p}_o \in \mathbb{N}$ be the total number of positions that comprises the finite priority sequence of object o, $\tau_o : \{1, \dots, \bar{p}_o\} \to N$ be a mapping from positions to agents such that no consecutive positions are assigned the same agent, i.e. for each $p \in \{1, \dots, \bar{p}_o - 1\}$, $\tau_o(p) \neq \tau_o(p+1)$, and $q_o : \{1, \dots, \bar{p}_o\} \to (0, 1]$ be a trading rights list for o specifying for each position an amount such that $\sum_{p \in \{1,\dots,\bar{p}_o\}} q_o(p) = 1$. Finally, let (q, τ) be such that for each $i \in N$, $\sum_{o \in O} \sum_{p \in \{1,\dots,\bar{p}_o\}:\ \tau_o(p)=i} q_o(p) = 1$.

We refer to each pair (q, τ) as a Generalized TPAC parameter. For each pair (q, τ), let $\varphi^{(q,\tau)}$ be the associated Generalized TPAC rule. For each $R \in \mathcal{R}^N$, we use the following algorithm to compute $\varphi^{(q,\tau)}(R)$:

> **Step 1**: Construct a weighted, directed graph as follows: The set of vertices is the set of agents and objects. For each agent i, there is a directed edge with weight 1 to their most preferred object according to R_i. For each object o, agent $\tau_o(1)$ is the highest priority agent. Let there be a directed edge with weight $q_o(1)$ from o to $\tau_o(1)$. At least one cycle exists. For each cycle, there is an edge with minimum weight w among edges in the cycle. For each agent i in the cycle, 1) assign i this amount w of the object o for which there is an edge from i to o, and 2) decrease by this amount w agent i's trading right of the object o' in the cycle for which there is an edge from o' to i.
>
> At the end of each step, we record all trades, updated trading rights, and the current agent in each object's priority sequence.
>
> Let $s \geq 1$. At the end of Step s, let $t^s = (t^s_{io})_{i \in N, o \in O} \in [-1, 1]^{N \times O}$ record the profile of trades executed, for each $o \in O$, $\hat{q}^s_o : \{1, \dots, \bar{p}_o\} \to [0, 1]$ be the updated trading rights, and $p^s = (p^s_o)_{o \in O} \in \prod_{o \in O} \{1, \dots, \bar{p}_o\}$ be the updated positions where: If $i \in N$ is not in a cycle at Step s, then for each $o \in O$, $t^s_{io} = 0$, and if in addition $i = \tau_o(p^{s-1}_o)$ (where for each $o \in O$, $p^0_o = 1$), then $p^s_o = p^{s-1}_o$, and
>
> $$\hat{q}^s_o(p) = \begin{cases} 0 & \text{if } p < p^s_o, \text{ and} \\ q^{s-1}_o(p) & \text{if } p \geq p^s_o, \end{cases}$$
>
> (where $\hat{q}^0_o = q_o$). If $i \in N$ is in a cycle at Step s, then let $o \in O$ be the object such that there is an edge from i to o, $o' \in O$ be the object in the cycle such that there is an edge from o' to i; if $o = o'$, then $t^s_{io} = 0$, and otherwise, let t^s_{io} be the minimum weight among all edges in the cycle, and $t^s_{io'} = -t^s_{io}$. In addition, there are two subcases to when i is in a cycle: $-t^s_{io'} = \hat{q}^{s-1}_{o'}$ and $-t^s_{io'} < \hat{q}^{s-1}_{o'}$. If $-t^s_{io'} = \hat{q}^{s-1}_{o'}$, then i has exhausted their

one without initial endowments and hence features no element of trading between the agents.

trading right of o' at position $p_{o'}^{s-1}$, so $p_{o'}^{s} = p_{o'}^{s-1} + 1$, $i \neq \tau_o(p_{o'}^{s})$, and

$$\hat{q}_{o'}^{s}(p) = \begin{cases} 0 & \text{if } p < p_{o'}^{s}, \text{ and} \\ q_{o'}^{s-1}(p) & \text{if } p \geq p_{o'}^{s}. \end{cases}$$

If $-t_{io'}^{s} < \hat{q}_{o'}^{s-1}$, then i has not exhausted their trading right of o' at position $p_{o'}^{s-1}$, so $p_{o'}^{s} = p_{o'}^{s-1}$, and $\hat{q}_{o'}^{s}$

$$\hat{q}_{o'}^{s}(p) = \begin{cases} 0 & \text{if } p < p_{o'}^{s}, \\ \hat{q}_{o'}^{s-1}(p) + t_{io'}^{s} & \text{if } p = p_{o'}^{s}, \text{ and} \\ q_{o'}^{s-1}(p) & \text{if } p > p_{o'}^{s}. \end{cases}$$

Step s: Construct a weighted, directed graph as follows: The set of vertices is the set of 1) agents i such that there is $o \in O$ and $p \geq p_o^{s-1}$ such that $\tau_o(p) = i$, and 2) objects o such that $\hat{q}_o^{s-1}(p^{s-1}) > 0$. For each agent i in the graph, there is a directed edge with weight $1 - \sum_{o \in O} \sum_{p < p_o^{s-1} : \tau_o(p) = i} q_o(p) - \sum_{o \in O : \tau_o(p_o^{s-1}) = i} q_o(p_o^{s-1}) - \hat{q}_o^{s-1}(p_o^{s-1})$ from i to their most preferred object in the graph according to R_i.[13] For each object o in the graph, there is a directed edge with weight $\hat{q}_o^{s-1}(p_o^{s-1})$ from o to $\tau_o(p_o^{s-1})$. At least one cycle exists. For each cycle, there is an edge with minimum weight w among edges in the cycle. For each agent i in the cycle, 1) assign i this amount w of the object o for which there is an edge from i to o, and 2) decrease by this amount w i's trading right of the object o' in the cycle for which there is an edge from o' to i .

The algorithm ends when there are no more agents. □

In comparison to the TPAC family, this generalization allows significantly more flexibility to alternate between agents' trading rights. Even so, for both families, the finiteness of the parameters precludes rules from satisfying full *sd-no-envy in net trade*. For small ϵ, it is impossible to achieve *ϵ-sd-no-envy in trade* within the TPAC family—if two agents compete for an object o by trading away object o', the fact that one agent always has trading rights of o' before another indicates a clear advantage to the prioritized agent.[14]

[13] This weight is 1 less the total amount of trading rights across all objects that i has used.

[14] For example, let $O = \{a, b, c\}$, $N = \{1, 2, 3\}$, $\omega = ((0.4, 0.4, 0.2), (0.4, 0.4, 0.2), (0.2, 0.2, 0.6))$, $R_1 = R_2 : abc$, and $R_3 : acb$. In the algorithm for each TPAC rule, each agent first consumes their own amount of a. Only b and c remain. If 1 has trading right of c before 2 does, then 1 and 3 eventually trade their c and b (and similarly so if 2 has an earlier trading right). Thus, any TPAC rule prescribes either $((0.4, 0.6, 0), (0.4, 0.4, 0.2), (0.2, 0, 0.8))$ or $((0.4, 0.4, 0.2), (0.4, 0.6, 0), (0.2, 0, 0.8))$, and for each $\epsilon < 0.2$, violates the *ϵ-sd-no-envy in net trade*.

We will show that it is possible to achieve arbitrarily closely *sd-no-envy in net trade* in our new family.

We first remark that *sd-efficiency* and the *sd-endowment lower bound* properties are inherited by this larger family.

Proposition 3. *Each Generalized TPAC rule is* sd-efficient.

Proposition 4. *The Generalized TPAC rule $\varphi^{(q,\tau)}$ satisfies the* sd-endowment lower bound *if and only if for each $i \in N$, and each $o \in O$, $\sum_{p\in\{1,\ldots,\bar{p}_o\}:\,\tau_o(p)=i} q_o(p) = \omega_{io}$.*

The proofs for the two statements are identical to Propositions 1 and 2, and we omit them here.

6.3 Sufficient Conditions for Fairness

We introduce a condition on the parameter (q, τ). Intuitively, it embodies the idea that no agent is ever "too far ahead" of another agent in terms of cumulative trading rights for each object. Let o be an object, and $p \in \{1, \ldots, \bar{p}_o\}$ be a position in o's priority sequence. Up until position p, we have alternated between various agents specifying who has a right to trade object o as well as how much. For a pair of agents i and j, we can compare the cumulative amount of trading rights each has exercised by the time we have arrived at position p. The condition requires that the difference between these cumulative amounts is bounded by ϵ, given that both agents have trading rights of o greater than these amounts. Put another way, no agent gets too far ahead of another in terms of trading rights.

We now present the condition formally. Let (q, τ) be a Generalized TPAC parameter, and for each $i \in N$, and each $o \in O$, $\bar{p}_{io} \in \{1, \ldots, \bar{p}_o\}$ be the last position in which i appears in τ_o.

Parameter (q, τ) satisfies **ϵ-no-ahead** if for each pair $i, j \in \{h \in N : \exists p \in \{1, \ldots, \bar{p}_o\}$ *s.t.* $\tau_o(p) = h\}$, each $o \in O$, and each $p' < \bar{p}_{io}$,

$$\epsilon + \sum_{p\le p':\,\tau_o(p)=i} q_o(p) \ge \sum_{p\le p':\,\tau_o(p)=j} q_o(p).$$

We now state the main result of this section: With the Generalized TPAC rules we can achieve arbitrarily closely our *sd-no-envy in net trade* fairness criterion. For each level of approximation, we provide sufficient conditions on the parameter (q, τ) for the associated Generalized TPAC rule to satisfy it. The condition permits a straightforward construction of a class of fair rules.

Theorem 2. *If (q, τ) satisfies ϵ-no-ahead, then $\varphi^{(q,\tau)}$ satisfies* $\epsilon|O|$-sd-no-envy in net trade.

We prove the theorem in Appendix B.

7 Conclusion

Trading probabilities along cycles is an intuitive extension of the successes in the discrete domain to the probabilistic. Our results demonstrate, though, that the difficulties between efficiency and manipulability appear. If there is even one agent who owns positive share of two different objects, then none of our *sd-efficient* rules satisfies the *sd-endowment lower bound* and *sd-strategy-proofness* together; only in the special case of discrete endowments do we have compatibility.

On the other hand, we show that an interesting notion of fairness is possible. In the classic exchange economy, *no-envy in net trade* essentially characterizes the Competitive Equilibrium allocations. To our knowledge, our paper is the first to consider the property in the probabilistic domain, and we establish sufficient conditions on our rule's parameters to arbitrarily closely achieve it.

We state some open questions. While we show that there are rules satisfying *sd-efficiency*, the *sd-endowment lower bound*, and *ϵ-sd-no-envy in net trade*, can the same be shown when the latter is replaced with *sd-no-envy in net trade*? Next, our adjustment of net trades guarantees agents' comparisons are well-defined, but a weaker adjustment is possible and results in a strengthening of the *sd-no-envy in net trade*. This naturally posits the question of whether there are mechanisms satisfying this stronger notion. Finally, our algorithms and those of Yu and Zhang (2021) are procedurally significantly different, obfuscating their relationship. Future work may clarify the precise relationship between the two families.

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Appendix A: Proof of Theorem 1

We first state several useful lemmas. They show that certain classes of TPAC parameters (r, σ) result in manipulable rules. We then show a special case of Theorem 1 wherein some agent owns three or more objects. Lastly, we finish with the proof of Theorem 1.

Lemma 1. *Let (r, σ) be a TPAC parameter. If there are $i, j \in N$ and $a, b, c \in O$ such that*

1. $r_{ia}, r_{ic} > 0$,
2. $r_{ja}, r_{jb} > 0$, *and*
3. σ *satisfies either below:*
   (a) $\sigma_c(i) = 1$ *and* $\sigma_b(j) = 1$, *or*
   (b) $\sigma_c(i) = \sigma_b(i) = 1$, *and* $\sigma_b(j) = 2$,

then $\varphi^{(r,\sigma)}$ is not sd-strategy-proofness.

Proof. Let $i = 1$, $j = 2$, and (a) be true. The proof for (b) is similar. Let $R \equiv (R_1, R_{-1}) \in \mathcal{R}^N$ and $R' \equiv (R'_1, R_{-1}) \in \mathcal{R}^N$ be as follows. [15]

R_1	R_2	$R_{-\{1,2\}}$	R'_1	R_2	$R_{-\{1,2\}}$
b	a	a	a	a	a
a	c	b	b	c	b
c	b	c	c	b	c
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

We compute the total amount of a and b that agent 1 receives at these two preference profiles.

First, we compute $\varphi^{(r,\sigma)}_{1a}(R) + \varphi^{(r,\sigma)}_{1b}(R)$. By *sd-efficiency*, $\varphi^{(r,\sigma)}_{1a}(R) = 0$. Thus, the question reduces to how much of b does 1 get? In the first step of the algorithm, agent 1 points to b and each other agent points to a. Note that this remains the same as long as a and b are not exhausted. From the first step until the step that object a is exhausted, each cycle falls into one of the following three cases:

> Case 1: *a points to $j \in N \setminus \{1\}$.* The only cycle is formed by $\{a, j\}$. Since j points to a with weight 1, and a points to j with weight r_{ja}, agent j receives r_{ja} amount of a. On the other hand, agent 1 receives nothing.

[15] For each $i, j \in N$, $R_{-\{i,j\}} = abc\ldots$ denotes the profile obtained from $R \in \mathcal{R}^N$ by deleting its i-th component R_i and j-th component R_j where for each $k \in N \setminus \{i, j\}$, we have $R_k = abc\ldots$.

Case 2: *a points to 1, and b points to* $j \in N\backslash\{1\}$. The only cycle is formed by $\{a, 1, b, j\}$, Since $\sum_{o\in O} r_{io} = 1$, $\sum_{i\in N} r_{io} = 1$, and $r_{1c} > 0$, we have

$$r_{1a} < 1 - r_{1b} = \sum_{j\in N\backslash\{1\}} r_{jb}.$$

Therefore, 1 trades a with various agents in return for b until they receive r_{1a} amount of b.

Case 3: *b points to* 1. The only cycle is formed by $\{b, 1\}$. Since agent 1 points to b with weight at least $1 - r_{1a} > r_{1b}$ and b points to 1 with weight r_{1b}, agent 1 receives r_{1b} amount of b.

Note that at any particular step, multiple cycles may occur (specifically a Case 1 cycle and a Case 2 cycle), but each cycle nevertheless falls into one of the three cases. Since $r_{1a} < 1 - r_{1b} = \sum_{j\in N\backslash\{1\}} r_{jb}$, object a is exhausted first. After a is exhausted, each agent $j \in N\backslash\{2\}$ points to b and 2 points to c. Similarly as before, the following three cases occur in different orders.

Case 1′: *b points to* $j \in N\backslash\{2\}$. The only cycle is formed by $\{b, j\}$. A special case where $j = 1$ is analyzed above in Case 3.

Case 2′: *b points to* 2*, and c points to* $j \in N\backslash\{2\}$. The only cycle is formed by $\{b, 2, c, j\}$.

Case 3′: *c points to 2.* The only cycle is formed by $\{c, 2\}$, and therefore agent 1 receives nothing.

We see that agent 1 receives $r_{1a} + r_{1b}$ amount of b. Do they get any positive amount of b from the trade that occurs in Case 2′?

First, we have to determine how much of trading right for b that 2 has when Case 2′ occurs. In Case 2, agent 2 may have traded some of their trading right for object b with agent 1 in return for object a. Moreover, by the time $\{a, 1, b, 2\}$ forms a cycle in Case 2, agent 1 may have traded their trading rights for object a with agents having a higher priority for b than agent 2 in return for object b. That is, by the time this occurs, agent 1 will have had

$$\max\left\{0, r_{1a} - \sum_{j\in N\backslash\{1\}:\, \sigma_b(j)<\sigma_b(2)} r_{jb}\right\} = r_{1a}$$

amount of trading right for object a left. The equality holds from the fact that $\sigma_b(2) = 1$ (or in the case of (b), $\sigma_b(1) = \sigma_b(2) - 1 = 1$). Then, in case 2, agent 2 trades their trading right for b with agent 1's remaining share of object a. Therefore, agent 2 has

$$\max\left\{0, r_{2b} - \max\{0, r_{1a} - \sum_{j\in N\setminus\{1\}:\, \sigma_b(j)<\sigma_b(2)} r_{jb}\}\right\} = \max\left\{0, r_{2b} - r_{1a}\right\}$$

amount of trading right left for object b.

By the time $\{b, 2, c, 1\}$ forms a cycle in Case 2′, agent 2 may have traded his remaining trading right for object b with agents having a higher priority for c than agent 1 in return for object c. That is, by the time this occurs, agent 2 will have had

$$\max\left\{0, \max\left\{0, r_{2b} - \max\{0, r_{1a} - \textstyle\sum_{j\in N\setminus\{1\}:\, \sigma_b(j)<\sigma_b(2)} r_{jb}\}\right\} - \textstyle\sum_{j\in N\setminus\{2\}:\, \sigma_c(j)<\sigma_c(1)} r_{jc}\right\}$$
$$= \max\left\{0, r_{2b} - r_{1a}\right\}$$

amount of trading right for object b left. Finally, during this case, agent 1 receives

$$\min\left\{r_{1c}, \max\left\{0, \max\left\{0, r_{2b} - \max\{0, r_{1a} - \textstyle\sum_{j\in N\setminus\{1\}:\, \sigma_b(j)<\sigma_b(2)} r_{jb}\}\right\} - \textstyle\sum_{j\in N\setminus\{2\}:\, \sigma_c(j)<\sigma_c(1)} r_{jc}\right\}\right\}$$
$$= \min\left\{r_{1c}, \max\{0, r_{2b} - r_{1a}\}\right\} \tag{1}$$

amount of b.

Since $r_{2b} \leq 1 - r_{2c} = \sum_{j\in N\setminus\{2\}} r_{jc}$, the total trading right of c of agents pointing to b is greater than the 2's trading right of b, and agent 1 cannot obtain any more b from 2. Furthermore, since each $j \in N\setminus\{1, 2\}$ receives their own r_{jb}, 1 cannot get any more b. To summarize, $\varphi^{(r,\sigma)}_{1a}(R) + \varphi^{(r,\sigma)}_{1b}(R)$ is equal to $r_{1a} + r_{1b} + (1)$.

Next, we compute $\varphi^{(r,\sigma)}_{1a}(R'_1, R_{-1}) + \varphi^{(r,\sigma)}_{1b}(R'_1, R_{-1})$. In the first part of the algorithm, each agent $j \in N$ points to a with weight 1, and a successively points to agents according to σ_a, for each $j \in N$, with weight r_{ja}. Thus, each agent $j \in N$, receives r_{ja} amount of a. Object a is exhausted after each agent receives their share of a. Then, in the next stage of the algorithm, each $j \in N\setminus\{2\}$ points to b with weight $1 - r_{ja}$ and 2 points to c with weight r_{2c}. From the step after a is exhausted until the step that object b is exhausted, each cycle falls into one of the following three cases:

Case 1″: *b points to $j \in N\backslash\{2\}$*. The only cycle is formed by $\{b, j\}$. Since $j \in N \setminus \{2\}$ points to b with weight at least $1 - r_{ja} \geq r_{jb}$, and j points to b with r_{jb}, agent j receives r_{jb} amount of b. In particular, agent 1 receives r_{1b} amount of b.

Case 2″: *b points to 2, and c points to $j \in N\backslash\{2\}$*. The only cycle is formed by $\{b, 2, c, j\}$.

Case 3″: *c points to 2*. The only cycle is formed by $\{c, 2\}$, in which agent 2 receives r_{2c} amount of c. Thus, agent 1 receives nothing.

We see that agent 1 receives r_{1a} amount of a, and r_{1b} amount of b. Do they get any positive amount of b from the trade that occurs in Case 2″?

First, we have to determine how much of trading right for b that agent 2 has when $\{b, 2, c, 1\}$ forms a cycle in Case 2″. By the time this occurs, agent 2 may have traded their trading rights for object b with agents having a higher priority for object c than agent 1 in return for object c. Therefore, by this time, agent 2 has

$$\max\left\{0, r_{2b} - \sum_{j\in N\backslash\{2\}:\, \sigma_c(j)<\sigma_c(1)} r_{jc}\right\} = r_{2b}$$

amount of trading right for object b left. Hence, in this case, agent 1 receives

$$\min\left\{r_{1c}, \max\left\{0, r_{2b} - \sum_{j\in N\backslash\{2\}:\, \sigma_c(j)<\sigma_c(1)} r_{jc}\right\}\right\} = \min\{r_{1c}, r_{2b}\} \tag{2}$$

amount of b. Since $r_{2b} \leq \sum_{j\in N\backslash\{2\}} r_{jc}$ and each $j \in N\backslash\{1,2\}$ receives r_{jb} amount of b, 1 cannot get anymore of b. To summarize, $\varphi^{(r,\sigma)}_{1a}(R'_1, R_{-1}) + \varphi^{(r,\sigma)}_{1b}(R'_1, R_{-1}) = r_{1a} + r_{1b} + (2)$.

By *sd-strategy-proofness*,

$$\varphi^{(r,\sigma)}_{1a}(R'_1, R_{-1}) + \varphi^{(r,\sigma)}_{1b}(R'_1, R_{-1}) = \varphi^{(r,\sigma)}_{1a}(R_1, R_{-1}) + \varphi^{(r,\sigma)}_{1b}(R_1, R_{-1}).$$

Equivalently, (1) is equal to (2).

If $r_{2b} - r_{1a} \leq 0$, then $(1) = 0 < \min\{r_{1c}, r_{2b}\} = (2)$. Thus, we have $r_{2b} - r_{1a} > 0$. Furthermore $r_{1c} \leq r_{2b} - r_{1a}$, otherwise $(1) < (2)$. Since $r_{1a} > 0$, we have $r_{1c} < r_{2b}$.

Note that both agents 1 and 2 own positive amount of a, and each owns positive amount of c and b, respectively. Consider another preference profile that permutes the roles of the agents 1 and 2, and objects b and c. Following symmetric reasoning, we can derive $r_{2b} < r_{1c}$. □

Lemma 2. *Let (r, σ) be a TPAC parameter. If there are $i, j \in N$, and $a, b, c \in O$ such that $\sigma_a(i) = \sigma_b(i) = \sigma_c(j) = 1$ and $r_{ia} + r_{ib} > r_{jc} > 0$, then $\varphi^{(r,\sigma)}$ is not* sd-strategy-proof.

Proof. Let $R \in \mathcal{R}^N$ be as below.

R_i	R_j	$R_{-\{i,j\}}$
c	a	a
a	b	c
b	c	$\vdots$
$\vdots$	$\vdots$	

Then, $\varphi^{(r,\sigma)}_{ia}(R)+\varphi^{(r,\sigma)}_{ic}(R) = r_{ia}+r_{ic}+\min\{r_{ib}, \max\{0, r_{jc}-r_{ia}\}\}$. If i reports $R'_i : acb\ldots$ instead, then $\varphi^{(r,\sigma)}_{ia}(R'_i, R_{-i}) + \varphi^{(r,\sigma)}_{ic}(R'_i, R_{-i}) = r_{ia} + r_{ic} + \min\{r_{ib}, r_{jc}\}$. By *sd-strategy-proofness*,

$$\min\{r_{ib}, \max\{0, r_{jc} - r_{ia}\}\} = \min\{r_{ib}, r_{jc}\}, \tag{3}$$

but this contradicts the hypothesis that $r_{1a} + r_{1b} > r_{2c}$. □

Lemma 3. *Let (r, σ) be a TPAC parameter. If there are $i, j \in N$ and $a, b \in O$ such that $r_{ia} + r_{ja} = r_{ib} + r_{jb} = 1$, then $\varphi^{(r,\sigma)}$ is not* sd-strategy-proof.

Proof. Without loss of generality, let $\sigma_b(i) = 1$. Let $R \in \mathcal{R}^N$ be as below.

R_i	R_j	$R_{-\{i,j\}}$
c	c	a
a	a	b
b	b	c
$\vdots$	$\vdots$	$\vdots$

Then, $\varphi^{(r,\sigma)}_{ia}(R) + \varphi^{(r,\sigma)}_{ic}(R) = r_{ia}$. If i reports $R'_i : acb\ldots$ instead, then $\varphi^{(r,\sigma)}_{ia}(R'_i, R_{-i}) + \varphi^{(r,\sigma)}_{ic}(R'_i, R_{-i}) = r_{ia} + \max\{1 - r_{ja}, r_{1b}\}$. Since $r_{ia}, r_{ib} > 0$, the maximum term is positive, and $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.* □

Proposition 5. *Let (r, σ) be a TPAC parameter. If there is an agent that owns positive trading rights of three or more objects, then $\varphi^{(r,\sigma)}$ is not* sd-strategy-proof.

We will make repeated use of Lemma 2, and write Lemma 2(i, j) if i and j are the relevant agents.

Proof. We will divide into three cases:

1. i is top priority in only one object, and middle priority in some object,
2. i is top priority in two or more objects, and middle priority in some object, and

3. i is top priority in all objects in which they have positive probability,

Let $O^i \subset O$ be the set of objects for which i owns positive trading right in r, $O^{i,top} \subset O^i$ be the set of objects for which i is top priority, $O^{i,mid} \subset O^i$ be the set of objects for which i is not top priority, $O^{-i} \subset O$ be the set of objects for which i does not own any trading right, and $a \in O^i$ be such that $\sigma_a(i) = 1$.

Case 1: For each $b \in O^i \backslash \{a\}$, $\sigma_b(i) \neq 1$. Relabel $O^i \backslash \{a\}$ as $b_1, \ldots, b_x$. By construction, there is an agent that is top priority at b_1 that is not i. Let j_1 be this agent. If there is b_2, then similarly, there is j_2. Note that $j_1 \neq j_2$, by Lemma 2(i, j_1). Continuing in this fashion, there is a sequence of objects $j_1, \ldots, j_x$ who are each top priority respectively at $b_1, \ldots, b_x$.

$$\sigma\text{:}\quad \begin{array}{cccc} a & b_1 & \cdots & b_x \\ \hline i & j_1 & & j_x \\ \vdots & \vdots & & \vdots \\ & i & \cdots & i \\ & \vdots & & \vdots \end{array}$$

Consider j_1. Since j_1 owns total amount 1 of trading rights, there is $c_1 \in O \backslash \{b_1\}$ such that $r_{j_1 c_1} > 0$. By Lemma 2(i, j_1), $c_1 \notin \{b_2, \ldots, b_x\}$. So $c \in O^{-i} \cup \{a\}$. Let $c_1 \in O^{-i}$ (we will cover the general case). Similarly, there is c_2 such that $r_{j_2 c_2} > 0$ and $c_2 \in O^{-i}$. By Lemma 2(j_1, j_2), $c_1 \neq c_2$. Repeating this reasoning, we have $c_1, \ldots, c_x$ where $j_1, \ldots, j_x$ owns positive trading rights of their respective object. For each $j_z \in \{j_1, \ldots, j_x\}$, Lemma 2$(i, j_z)$ implies that $\sigma_{c_z}(j_z) \neq 1$ (shown below).

$$\sigma\text{:}\quad \begin{array}{ccccccc} a & b_1 & \cdots & b_x & c_1 & \cdots & c_x \\ \hline i & j_1 & & j_x & k_1 & & k_x \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ & i & & i & j_1 & & j_x \\ & \vdots & & \vdots & \vdots & & \vdots \end{array}$$

Consider c_1. Let $k_1 \in N \backslash \{i, j_1\}$ be such that $\sigma_{c_1}(k_1) = 1$. For each $j_z \in \{j_2, \ldots, j_x\}$, Lemma 2$(j_1, j_z)$ implies that $k_1 \neq j_z$. So $k_1 \notin \{i, j_1, \ldots, j_x\}$. Repeating the reasoning as in the previous paragraph, there is $k_1, \ldots, k_x \in N \backslash \{i, j_1, \ldots, j_x\}$ such that each is the top priority respectively at $c_1, \ldots, c_x$ (shown above) and for each $k_y, k_z \in \{k_1, \ldots, k_x\}$, $k_y \neq k_z$.

Since each $k_z \in \{k_1, \dots, k_x\}$ owns total amount 1 of trading right, there is $d_1, \dots, d_x \in O \backslash \{b_1, \dots, b_x\}$ such that $k_1, \dots, k_z$ respectively have positive trading right of the object. Furthermore, for each $k_y, k_z \in \{k_1, \dots, k_x\}$, by Lemma $2(k_y, k_z)$, $d_y \neq d_z$ and $d_y \neq c_z$. So $d_1, \dots, d_x \in O \backslash \{b_1, \dots, b_x, c_1, \dots, c_x\}$.

Thus, we are able to construct a sequence $i, j_1, \dots, j_x, k_1, \dots, k_x, ...$ with the aforementioned properties. By finiteness of N, there is some agent ℓ^* such that ℓ^* is at the top priority of some object, and by feasibility of r and finiteness of O, $r_{\ell^* a} > 0$ (shown below).

$$
\begin{array}{r|ccccccccccccc}
\sigma\!: & a & b_1 & \cdots & b_x & c_1 & \cdots & c_x & d_1 & \cdots & d_x & \cdots & e \\
\hline
 & i & j_1 & & j_x & k_1 & & k_x & & & & & \ell^* \\
 & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\
 & \ell^* & i & & i & j_1 & & j_x & k_1 & & k_x & & \\
 & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots & &
\end{array}
$$

Since i owns three or more objects, $x > 1$. We can thus continue to construct the sequence as above with the other $x - 1$ agents, contradicting the finiteness of N.

Case 2: There is $a', b \in O^i \backslash \{a\}$ such that $\sigma_{a'}(i) = 1$, and $\sigma_b(i) \neq 1$. We construct a sequence of agents as in Case 1. Note that each $j_1, \dots, j_x$ as above now cannot have positive trading right of a or a' or any $a'' \in \{\hat{a} \in O^i : \sigma_{\hat{a}}(i) = 1\}$, by Lemma $2(i, j_1)$, so this sequence of agents must be infinite—contradicting the finiteness of N.

Case 3: For each $x \in O^i$, $\sigma_x(i) = 1$.

Case 3.1: $O^i = O$. Consider j who is second priority after 1 for some object. By definition of r, j appears in some other object. By Lemma 2, $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.*

Case 3.2: $O^i \neq O$, & $\exists j \in N \backslash \{i\}$ such that $|O^j| \geq 2$ and for each $x \in O^j$, $\sigma_x(j) = 1$. By Lemma 3, $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.*

Case 3.3: $O^i \neq O$, & $\forall j \in N \backslash \{i\}$ such that $|O^j| \geq 2$, there is $x \in O^j$ such that $\sigma_x(j) \neq 1$. Consider $x \in O \backslash O^i$, and let $j \in N \backslash \{i\}$ be the top priority agent at x. If $|O^j| \geq 3$, then by Case 1 and Case 2, we are done. Thus, for each $x \in O \backslash O^i$, the top priority agent owns trading right of either one or two objects.

Let $b_1 \in O \backslash O^i$. If there is only one owner of trading rights of b_1, then this agent owns full trading right of b_1 (Case 3.3.1). Instead, let there be at least two owners. Let $j_1 \in N\backslash\{i\}$ be the top priority agent at b_1.

By definition of r and Case 3.3, there is $b_2 \in O\backslash(O^i \cup \{b_1\})$ such that $r_{b_2 j_1} > 0$ and $\sigma_{b_2}(j_1) \neq 1$. Let $j_2 \in N\backslash\{i, j_1\}$ be such that $\sigma_{b_2}(j_2) = 1$. Observe that j_2 owns positive trading right of one other object in $O\backslash(O^i \cup \{b_2\})$. Let b_3 be this object. Continuing in this manner, by finiteness of O, we can construct a sequence of agents $j_1, \ldots, j_x$ and objects $b_1, \ldots, b_x$ such that $\sigma_{b_1}(j_1) = 1 \neq \sigma_{b_2}(j_1)$,..., and $\sigma_{b_x}(j_x) = 1$ and $r_{b_1 j_x} > 0$ (shown below).

$$\sigma\colon \begin{array}{ccccc} a & \cdots & b_1 & \cdots & b_x \\ \hline i & \cdots & j_1 & & j_x \\ \vdots & & j_x & \cdots & j_{x-1} \end{array}$$

Futhermore, for each $b_1, \ldots, b_x$, there are only two owners: if there is $h \in N\backslash\{i, j_1, \ldots, j_x\}$ such that h owns positive trading right of one of $b_1, \ldots, b_x$, then there is some agent in $j_1, \ldots, j_x$ that owns positive trading right of a third object—contradicting the definition of Case 3.3.

Case 3.3.1: $x \geq 3$. The following economy shows that $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.*

$$\sigma: \begin{array}{ccc} b_1 & b_2 & b_3 \\ \hline j_1 & j_2 & j_3 \\ \vdots & j_1 & j_2 \end{array} \qquad \begin{array}{ccc} R_{j_1} & R_{j_2} & R_{j_3} \\ \hline b_3 & b_2 & b_3 \\ b_2 & b_1 & \vdots \\ b_1 & \vdots & \\ \vdots & & \end{array} \qquad \begin{array}{c} R'_{j_1} \\ \hline b_2 \\ b_3 \\ b_1 \\ \vdots \end{array}$$

By above, either $r_{j_1 b_1} = r_{j_2 b_2} = r_{j_3 b_3} \geq r_{j_1 b_2} = r_{j_2 b_3}$ or $r_{j_1 b_1} = r_{j_2 b_2} = r_{j_3 b_3} < r_{j_1 b_2} = r_{j_2 b_3}$. In the former case, j_1 gets $r_{j_1 b_2}$ of b_2 and b_3 in the truth, and $r_{j_1 b_2} + r_{j_2 b_3}$ in the lie. In the latter case, j_1 gets the same as in the truth, and $r_{j_1 b_2} + r_{j_1 b_1}$ of b_2 and b_3 in the lie.

Case 3.3.2: $x = 2$. By Lemma 4, $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.*

Case 3.3.3: There is one agent that owns b_1. By the reasoning of Cases 3.3.2 and 3.3.3, each agent $j \in N\backslash\{i\}$ that has positive trading right of some object $x \in O\backslash O^i$, has full trading right of x. Thus, each $j \in N\backslash\{i\}$ such that j owns positive trading right of some object in O^i is such that $O^j \subseteq O^i$. Let $j \in N\backslash\{i\}$ be such that j is second priority for some object in O^i. Since $|O^i| \geq 3$, by Lemma 2(i,j), $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.* $\square$

Proof of Theorem 1. By Proposition 5, each agent owns positive probability of either one or two objects. Suppose by contradiction that there is $i \in N$, and $a, b \in O$ such that $r_{ia}, r_{ib} > 0$ and $\sigma_a(i) = 1$.

Let $\sigma_b(i) = 1$. By Lemma 2, for each $c \in O\backslash\{a,b\}$, there is one agent $k \in N\backslash\{i\}$ that owns full probability of c; otherwise, $r_{ia} + r_{ib} = 1 > r_{kc}$ (if k is the first priority agent of c). By definition of r, there is $j \in N\backslash\{i\}$ with $r_{ja} + r_{jb} = 1$. By Lemma 3, $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.*

Let $\sigma_b(i) = 2$. By definition of r, there is $j_1 \in N\backslash\{i\}$ such that $r_{jb} = 1 - r_{ib}$. If j has positive trading right of b, then by Lemma 3, $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.* Thus, j has positive trading right of some $c \in O\backslash\{a,b\}$. By Lemma 1, $\sigma_c(j_1) \neq 1$, so there is $j_2 \in N\backslash\{1, j_1\}$ such that $\sigma_c(j_2) = 1$. Finally, the example in Case 3.3.1 of Proposition 5 shows that $\varphi^{(r,\sigma)}$ is not *sd-strategy-proof.* $\square$

Appendix B

Claim 1. Let x be an allocation, and $i, j \in N$. Then, $x_i^j \in \triangle O$.

Proof. Note that for each $j \in N$, $\sum_{o\in O} t_{jo} = 0$. Thus, for each $i, j \in N$, $\sum_{o \in O} x_{io}^j = \sum_{o\in O} \omega_{io} + \alpha(\omega_i, \omega_j) \sum_{o\in O} t_{jo} = \sum_{o\in O} \omega_{io} = 1$.

Let $j \in N$ and $a \in O$ be such that $t_{ja} < 0$, and therefore, $|t_{ja}| \leq \omega_{ja}$. Clearly, $x_{ia}^j \leq 1$. Suppose on the contrary that $x_{ia}^j = \omega_{ia} + \alpha(\omega_i, \omega_j) t_{ja} < 0$. That is, $\omega_{ia} < \alpha(\omega_i, \omega_j)|t_{ja}| \leq \alpha(\omega_i, \omega_j)\omega_{ja}$, which implies $\frac{\omega_{ia}}{\omega_{ja}} < \alpha(\omega_i, \omega_j)$, a contradiction. Therefore, $x_{ia}^j \in [0,1]$.

Let $j \in N$ and $a \in O$ be such that $t_{ja} > 0$. Clearly, $x_{ia}^j \geq 0$. Suppose on the contrary that $x_{ia}^j = \omega_{ia} + \alpha(\omega_i, \omega_j) t_{ja} > 1$. Since, $\sum_{o\in O} x_{io}^j = 1$, there is $b \in O \setminus \{a\}$ such that $x_{ib}^j < 0$, a contradiction.

Hence, for each $i, j \in N$ and each $a \in O$, $x_{ia}^j \in [0,1]$ and $\sum_{o\in O} x_{io}^j = 1$, which imply $x_i^j \in \triangle O$. $\square$

Proof of Thereom 2. Let $R \in \mathcal{R}^N$, (q, τ) be a Generalized TPAC parameter that satisfies ϵ-no-ahead, and $x \equiv \varphi^{(q,\tau)}(R)$. Let $o \in O$, $i, j \in N$, and $B \subseteq O$ be such that $B \equiv \{b \in O : bR_i o\}$. Let (t_{io}^k) be the trade vector generated from the algorithm for computing $\varphi^{(q,\tau)}(R)$. For each subset $B \subseteq O$, let t_{iB}^k be the total accumulation of objects in B at the end of Step k, that is, $t_{iB}^k \equiv \sum_{o\in B} t_{io}^k$.

We want to show that

$$\epsilon + \sum_{b\in B} x_{ib} \geq \sum_{b\in B} x^j_{ib}.$$

Step 1: Express the summations in the inequality above in terms of objects in $O\backslash B$.

By definition of x_i and x^j_i, the inequality becomes

$$\epsilon + \sum_{b\in B} (\omega_{ib} + t_{ib}) \geq \sum_{b\in B} (\omega_{ib} + \alpha(i,j)t_{jb})$$

and simplifying,

$$\epsilon + \sum_{b\in B} t_{ib} \geq \sum_{b\in B} \alpha(i,j)t_{jb}.$$

Let S be the last step of the algorithm for computing $\varphi^{(\sigma,r)}(R)$, and $s(B)$ be the last step that an object in B is still available. By definition of $s(B)$, the inequality becomes

$$\epsilon + \sum_{s\leq s(B)} \sum_{b\in B} t^s_{i,b} \geq \sum_{s\leq s(B)} \sum_{b\in B} \alpha(i,j)t^s_{j,b}. \tag{4}$$

By definition of t, we rewrite components of the LHS then RHS of equation (4) respectively as:

$$\begin{aligned}
&\textstyle\sum_{b\in B} t^s_{ib}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{ib}=0\}} t^s_{ib} + \sum_{\{b\in B:\, t^s_{ib}\neq 0\}} t^s_{ib}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{ib}>0 \text{ and } \exists d\notin B,\, t^s_{id}<0\}} t^s_{ib} + \sum_{\{b\in B:\, t^s_{ib}>0 \text{ and } \exists d\in B,\, t^s_{id}<0\}} t^s_{ib}\\
&\quad \textstyle+ \sum_{\{b\in B:\, t^s_{ib}<0 \text{ and } \exists d\in B,\, t^s_{id}>0\}} t^s_{ib}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{ib}>0 \text{ and } \exists d\notin B,\, t^s_{id}<0\}} t^s_{ib}\\
&= \textstyle\sum_{\{d\notin B:\, \exists b\in B,\, t^s_{ib}>0 \text{ and } t^s_{id}<0\}} |t^s_{id}|
\end{aligned}$$

$$\begin{aligned}
&\textstyle\sum_{b\in B} t^s_{jb}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{jb}=0\}} t^s_{jb} + \sum_{\{b\in B:\, t^s_{jb}\neq 0\}} t^s_{jb}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{jb}>0\}} t^s_{jb} + \sum_{\{b\in B:\, t^s_{jb}<0\}} t^s_{jb}\\
&= \textstyle\sum_{\{b\in B:\, t^s_{jb}>0 \text{ and } \exists d\notin B,\, t^s_{jd}<0\}} t^s_{jb} + \sum_{\{b\in B:\, t^s_{jb}>0 \text{ and } \exists d\in B,\, t^s_{jd}<0\}} t^s_{jb}\\
&\quad \textstyle+ \sum_{\{b\in B:\, t^s_{jb}<0 \text{ and } d\in B,\, t^s_{jd}>0\}} t^s_{jb} + \sum_{\{b\in B:\, t^s_{jb}<0 \text{ and } d\notin B,\, t^s_{jd}>0\}} t^s_{jb}\\
&= \textstyle\sum_{\{d\notin B:\, \exists b\in B,\, t^s_{jb}>0 \text{ and } t^s_{jd}<0\}} |t^s_{jd}| + \sum_{\{d\notin B:\, \exists b\in B,\, t^s_{jb}<0 \text{ and } t^s_{jd}>0\}} -|t^s_{jd}|\\
&\leq \textstyle\sum_{\{d\notin B:\, \exists b\in B,\, t^s_{jb}>0 \text{ and } t^s_{jd}<0\}} |t^s_{jd}|
\end{aligned}$$

Since $\alpha(i,j) \in [0,1]$, we can compare the RHS to the new expression:

$$\sum_{s \leq s(B)} \sum_{b \in B} \alpha(i,j) t^s_{jb} \leq \sum_{s \leq s(B)} \sum_{\{d \notin B : \, \exists b \in B, \, t^s_{jb} > 0 \text{ and } t^s_{jd} < 0\}} \alpha(i,j) |t^s_{jd}|.$$

Thus, it is sufficient to show that

$$\epsilon + \sum_{s \leq s(B)} \sum_{\{d \notin B : \, \exists b \in B, \, t^s_{ib} > 0 \text{ and } t^s_{id} < 0\}} |t^s_{id}| \geq \sum_{s \leq s(B)} \sum_{\{d \notin B : \, \exists b \in B, \, t^s_{jb} > 0 \text{ and } t^s_{jd} < 0\}} \alpha(i,j) |t^s_{jd}| \qquad (5)$$

Step 2: Use ϵ-*no-ahead* to show the desired inequality for each object $d \in O \setminus B$.

Let $d \in O \setminus B$. Then, for each $b \in B$, $b \; P_i \; d$, and at each Step $s \leq s(B)$, agent i either trades d for another object in B or does not use d. Thus, for each $s \leq s(B)$, $t^s_{id} \leq 0$.

Case 1: At the end of Step $s(B)$, agent i has a positive remaining trading rights of object d. That is,

$$\sum_{s \geq s(B) : \tau_d(p^s_d) = i} \hat{q}_d(p^s_d) > 0.$$

This also implies that $p^{s(B)}_d \leq \bar{p}_{id}$.

Case 1.1: It is not agent i's turn to trade object d at Step $s(B)$, i.e. $\tau_d(p^{s(B)-1}_d) \neq i$. Since at each Step $s \leq s(B)$, agent i either trades d for another object in B or does not use d, we have

$$\sum_{\{p \leq p^{s(B)}_d : \tau_d(p) = i\}} q_d(p) = \sum_{s \leq s(B)} |t^s_{id}|.$$

Moreover, since $p^{s(B)}_d \leq \bar{p}_{id}$ and (q, τ) satisfies ϵ-*no-ahead*, for each $j \in N \setminus \{i\}$,

$$\epsilon + \sum_{\{p \leq p^{s(B)}_d : \tau_d(p) = i\}} q_d(p) \geq \sum_{\{p \leq p^{s(B)}_d : \tau_d(p) = j\}} q_d(p).$$

Combining these, we get

$$
\begin{aligned}
\epsilon + \sum_{s \le s(B)} |t^s_{id}| &\ge \sum_{\{p \le p_d^{s(B)}: \tau_d(p)=j\}} q_d(p) \\
&\ge \sum_{\{s \le s(B):\, t^s_{jd}<0\}} |t^s_{jd}| \\
&\ge \sum_{\{s \le s(B):\, t^s_{jd}<0 \text{ and } \exists b\in B,\, t^s_{jb}>0\}} |t^s_{jd}| \\
&\ge \alpha(i,j) \sum_{\{s \le s(B):\, t^s_{jd}<0 \text{ and } \exists b\in B,\, t^s_{jb}>0\}} |t^s_{jd}|,
\end{aligned}
$$

where the second inequality is due to the fact that at some step before $s(B)$, j may have consumed their own right of d, and the third inequality is due to the the fact that, additionally, j may have used their right of d to get some object outside of B.

Case 1.2: It is agent i's turn to trade object d at step $s(B)$, i.e. $\tau_d(p_d^{s(B)-1}) = i$. Now, since $p_d^{s(B)-1} - 1 < \bar{p}_{id}$ and (q,τ) satisfies ϵ-*no-ahead*, for each $j \in N \setminus \{i\}$,

$$
\epsilon + \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=i\}} q_d(p) \ge \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=j\}} q_d(p).
$$

Hence, we have

$$
\begin{aligned}
\epsilon + \sum_{s \le s(B)} |t^s_{id}| &= \epsilon + \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=i\}} q_d(p) + \sum_{\{s \le s(B):\, p_d^{s-1}=p_d^{s(B)-1}\}} |t^s_{id}| \\
&\ge \epsilon + \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=i\}} q_d(p) \\
&\ge \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=j\}} q_d(p) \\
&= \sum_{\{p \le p_d^{s(B)-1}-1:\, \tau_d(p)=j\}} q_d(p) + \sum_{\{p = p_d^{s(B)-1}:\, \tau_d(p)=j\}} q_d(p) \\
&= \sum_{\{p \le p_d^{s(B)-1}:\, \tau_d(p)=j\}} q_d(p) \\
&\ge \sum_{\{s \le s(B):\, t^s_{jd}<0\}} |t^s_{jd}| \\
&\ge \sum_{\{s \le s(B):\, t^s_{jd}<0 \text{ and } \exists b\in B,\, t^s_{jb}>0\}} |t^s_{jd}| \\
&\ge \alpha(i,j) \sum_{\{s \le s(B):\, t^s_{jd}<0 \text{ and } \exists b\in B,\, t^s_{jb}>0\}} |t^s_{jd}|.
\end{aligned}
$$

Case 2: Suppose that at the end of step $s(B)$, agent i does not have any positive remaining trading rights of object d. That is

$$
\sum_{\{s \ge s(B): \tau_d(p_d^s)=i\}} \hat{q}_d(p) = 0.
$$

In other words, agent i trades the amount of d they own in return for objects in B. Therefore,

$$\sum_{s \leq s(B)} |t^s_{i,d}| = \omega_{id}$$

If $\omega_{jd} > 0$, then since $\alpha(i,j) \leq \frac{\omega_{id}}{\omega_{jd}}$ and $\omega_{id} \geq \sum_{\{s \leq s(B):\, t^s_{jd} < 0 \text{ and } \exists b \in B,\, t^s_{jb} > 0\}} |t^s_{jd}|$, we have

$$\begin{aligned}
\epsilon + \textstyle\sum_{s \leq s(B)} |t^s_{id}| &= \epsilon + \omega_{id} \\
&\geq \alpha(i,j)\omega_{jd} \\
&\geq \alpha(i,j) \textstyle\sum_{\{s \leq s(B):\, t^s_{jd} < 0 \text{ and } \exists b \in B,\, t^s_{jb} > 0\}} |t^s_{jd}|.
\end{aligned}$$

If $\omega_{jd} = 0$, then notice that two expressions are zero, and the inequality holds trivially. This concludes Case 2.

Step 3: Combining both cases and sum over $d \in O \backslash B$. By Steps 1 and 2, we have

$$\begin{aligned}
&\epsilon|O| + \textstyle\sum_{\{s \leq s(B):\, \exists d \notin B,\, t^s_{id} < 0 \text{ and } \exists b \in B,\, t^s_{ib} > 0\}} |t^s_{id}| \\
\geq\ & \epsilon|O \backslash B| + \textstyle\sum_{\{s \leq s(B):\, \exists d \notin B,\, t^s_{id} < 0 \text{ and } \exists b \in B,\, t^s_{ib} > 0\}} |t^s_{id}| \\
=\ & \textstyle\sum_{d \notin B} \left(\epsilon + \textstyle\sum_{\{s \leq s(B):\, t^s_{id} < 0 \text{ and } \exists b \in B,\, t^s_{ib} > 0\}} |t^s_{id}| \right) \\
=\ & \textstyle\sum_{d \notin B} \left(\epsilon + \textstyle\sum_{s \leq s(B)} |t^s_{id}| \right) \\
\geq\ & \textstyle\sum_{d \notin B} \left(\alpha(i,j) \textstyle\sum_{\{s \leq s(B):\, t^s_{jd} < 0 \text{ and } \exists b \in B,\, t^s_{jb} > 0\}} |t^s_{jd}| \right) \\
=\ & \alpha(i,j) \textstyle\sum_{\{s \leq s(B):\, \exists d \notin B,\, t^s_{jd} < 0 \text{ and } \exists b \in B,\, t^s_{jb} > 0\}} |t^s_{j,d}|.
\end{aligned}$$

$\square$