

Increasing the Representation of a Targeted Group in a Reserve System

Abdullah Almeer Umut Dur Will Harris Greg Hauser
William Phan Yanning Zhang

April 6, 2022

Abstract

We study the assignment of an object with multiple copies where various numbers can be reserved for groups of individuals. We focus on the reserve system's ability to increase representation of a targeted group. We consider a general setting that covers many features considered in the recent literature; in particular, an individual can belong to more than one group and the priorities of individuals may differ across reserved copies. We show how a precedence order can be modified to increase the representation of the targeted group.

1 Introduction

Consider the problem of assigning homogeneous copies of an object (e.g., school seats, work permits, public sector jobs, or vaccines) to a set of individuals. Each individual receives at most one copy of the object and falls into one *or more* categories of relevant

types regarding age, gender, race, socioeconomic status, etc. Each copy can prioritize a particular type, but does not necessarily have to do so; in this case, we refer to them as *open* copies. If a copy of an object prioritizes type t individuals over the others, then we say that copy is *reserved* for type t . In this paper, we focus on the ability to increase representation of a particular type, while still respecting priority orders of the object copies.¹ We search for techniques that apply independently of and across all possible priority orders. This grants transparency—the selection process cannot be affected by the particular set of applicants considered and thus any acceptance changes are due purely to representation goals.

We consider a general model with the express purpose of capturing (almost) all of the key features of problems studied in the recent literature.² Since we allow individuals to have more than one type, a pair who shares types may reverse their relative priority across object copies. Similarly, subsets of individuals may overlap with various types while reversing orders in complex ways. Due to this generality, achieving greater representation of a certain type is not straightforward.

Our contribution is to shed light on how different ways of implementing a reserve system may influence the final allocation, above and beyond just the numbers reserved for each type. While reserve numbers are salient and easy to understand, the influences of other structures on welfare are less obvious. In particular, we consider the different *orders* in which we process the copies of the object. If the copies s_1, s_2, \dots, s_n of the object are processed sequentially in order of their index, then we first assign s_1 using the priority order associated with s_1 , then similarly with s_2 , and so on. We refer to this as *sequential processing*, and to the order that we follow as a *precedence order* on the set

¹An allocation respects priority orders if there does not exist individuals i and j such that j is assigned while i is not, but i has higher priority than j at the copy that j is assigned to.

²Section 2 details the models that we cover.

of copies (Kominers and Sönmez, 2016). We may observe significantly different welfare effects for different types depending on which precedence order is used. These effects are neglected if one simply considers the numbers of reserves.

We first give a characterization of a subset of allocations of the copies respecting priorities via sequential processing. For any precedence order, sequentially processing the copies results in an allocation that respects priorities (Proposition 1a). Consider the other direction. When the priority order for each copy ranking type t individuals is the same, and the analogous statement holds for every other type, any allocation respecting priorities can be achieved by sequential assignment for *some* precedence order (Proposition 1b).

Next we examine the effects of varying the precedence orders on the number of the assigned individuals with a targeted type. Due to the generality of the model, the precedence order that maximizes representation of individuals with a targeted type depends on priorities (Example 2).

We thus take a broad approach: for *any* priority order, we show how to adjust a given precedence order so that representation of individuals with the targeted type weakly increases. Let t^* be the targeted type, D_{t^*} be the set of all types t such that all individuals of type t are also of type t^* , and U_{t^*} be the set of all types t such that the set of type t and type t^* individuals are not connected (via some sequence of types). We show that, for any priority order, processing copies that prioritize individuals of types in D_{t^*} after all other copies increases the representation of type t^* individuals (Proposition 2). When all open copies are adjacent in the given precedence order, we show that the representation of type t^* individuals weakly increases by placing all copies that prioritize individuals of types in U_{t^*} before the open copies without changing the relative orders (Proposition 3). Finally, we show that when every type is either in D_{t^*}

or U_{t^*} , we can achieve greater representation of t^* individuals by moving all copies that prioritize individuals of types in U_{t^*} in front of all open copies, and all open copies in front of all copies that prioritize individuals of types in D_{t^*} (Proposition 4).

Our analysis reveals the importance of connectedness of different types as a constraint on the ability to increase representation by adjusting precedence orders. Many types may be connected simply by one individual. In such a case, assignment of that individual to a seat will change the connectedness of types when remaining individuals are considered. Therefore, our result might become suggestive for the assignment of remaining individuals.

The paper proceeds as follows: Section 2 discusses the relevant literature. In Section 3 we present our model, and in Section 4 we discuss our results. Section 5 concludes.

2 Literature Review

The growing literature on allocating objects when individuals have types and priorities stems from the importance of distributional goals reflecting fairness, representation, or public objectives in various centralized assignment problems. We highlight works closest to ours.

In school choice, diversity and affirmative action can be implemented with slot-specific priorities (where a slot is a seat at the school).³ Dur et al. (2018) study Boston’s school choice system in which each school reserves 50% of its seats for walk-zone students, and the other 50% are left open to all applicants. Their analysis highlights the effect of precedence orders on the outcome of the reserve system. Dur et al. (2020) analyze affirmative action policies in Chicago’s public schools and examine the balance between diversity and merit objectives. They show that, even in a “tier-blind” procedure, a group

³Kominers and Sönmez (2016) introduce slot-specific priorities in a many-to-one matching with contracts model, and demonstrate the wide applicability of this model in various matching markets including the cadet-branch matching and airline seat allocation problems.

can be favored (or not) due to correlations between tier and merit via the order in which slots are processed.

Sönmez and Yenmez (2019) consider the problem of allocating government jobs and college seats in India under *vertical* and *horizontal* reservations.⁴ They show that the choice rule mandated by the Indian Supreme Court is not well-defined, does not respect priorities, and is manipulable by hiding vertical categories. They propose choice rules that alleviate the issues above, and deviate minimally from the current rule in usage. Priority in their model stems from applicants’ “merit scores”, so two individuals of the same type will always maintain the same pairwise rank; in our environment, these two individuals may reverse in priority across different seats’ priority orders. In a model with just horizontal traits, Sönmez and Yenmez (2020) characterize choice rules that are non-wasteful, respecting priorities, and complying with reserves while accepting the maximal number of applicants. They do so for the case when an individual’s assignment counts for one reserve trait only, as well as the case where her assignment counts for all traits and individuals have at most two traits. We do not consider horizontal reservations in our model.

Pathak et al. (2020) examine the H1-B visa allocation program in the US. In their model, there are two types of applicants and positions—general-category and reserved-category—each with their own priority order. Among the reforms adopted in the last 15 years, they show that the latest reforms of 2019 ensure the largest number of reserved category applicants in the final allocation.

Motivated by the COVID-19 pandemic, Pathak et al. (2021) study reserve design in the allocation of medical resources such as ventilators and vaccines. They characterize the set of non-wasteful allocations which respects priorities as the set of all cutoff

⁴Aygün and Turhan (2020b) relatedly study affirmative action in Indian engineering colleges with vertical reservations only.

equilibria. They consider precedence orders over categories and show that the later a category is processed, the more individuals from that category are selected. Finally, they propose a “smart” reserve matching algorithm to alleviate issues of Pareto-inefficiency.

Our model includes almost all of the key features of those mentioned above. Table 1 summarizes these in the aforementioned papers and ours.⁵ Our contribution is to specify how we can increase representation of a targeted type with respect to a *given* precedence order, and can be applied in each of the other settings. In more restrictive environments, Dur et al. (2020), Pathak et al. (2020), and Pathak et al. (2021) provide the precedence order which maximizes representation of a targeted type. Our results do not prescribe the precedence order which maximizes representation of a targeted type. Instead, our results can be helpful to policymakers when they are faced with the more general problem and computing such a maximal precedence order is difficult.

	DPKS	DPS	SYa	SYb	PRJS	PSUY	This Paper
Types							
overlapping	-	-	+	+	-	+	+
Priority Order							
single	+	+	+	+	+	+	+
multiple	+	+	+	-	+(2)	+	+
Precedence							
over categories	+	+	-	-	+	+	+
over indiv. seats	+	+	-	-	-	-	+

Table 1: Comparing types of reserves and priorities in Dur et al. (2018), Dur et al. (2020), Sönmez and Yenmez (2019), Sönmez and Yenmez (2020), Pathak et al. (2020), Pathak et al. (2021), and this paper. Overlapping reserves indicate that an individual can have several different types, thus qualifying for seats reserved for each type.

⁵In addition to the literature summarized in the table, other related papers on reserves are Hafalir et al. (2013), Fragiadakis and Troyan (2017), Delacrétaz (2021), Aygün and Bó (2021), Aygün and Turhan (2020a), Sönmez and Yenmez (2021), and Pathak et al. (2022).

3 Model

We study the assignment of a finite set of (identical) **seats**, S , to a finite set of **individuals**, I . Each individual can receive at most one seat, is indifferent between each, and prefers having one to none.

Let T be a finite set of **types**. Types can be defined over the attributes of the individuals such as age, gender, and race. Let $\tau : I \rightrightarrows T$ be the type correspondence of individuals. We denote the set of individuals with type t as $I_t = \{i \in I : t \in \tau(i)\}$.⁶ Since an individual may have multiple types, it is possible that $I_t \cap I_{t'} \neq \emptyset$ for some $t, t' \in T$. If $I_t \subseteq I_{t'}$, then we say type t' **includes** type t . Let $D_{t'} = \{t \in T : I_t \subseteq I_{t'}\}$ be the set of types that type t' includes. We say types t and t' are **unrelated** if $I_t \cap I_{t'} = \emptyset$ and there does not exist a sequence of distinct types (t_1, \dots, t_k) such that $I_t \cap I_{t_1} \neq \emptyset$, $I_{t_1} \cap I_{t_2} \neq \emptyset, \dots, I_{t_{k-1}} \cap I_{t_k} \neq \emptyset$, and $I_{t_k} \cap I_{t'} \neq \emptyset$. Types t and t' are **related** if they are not unrelated. For each type t , we denote the related and unrelated types with R_t and U_t , respectively.⁷ Notice that if $t' \in R_t$, then $R_t = R_{t'}$ and $U_t = U_{t'}$.

Each seat s prioritizes individuals according to a strict order π^s over I . We say π^s is the **priority order** of seat s . Let $\pi = (\pi^s)_{s \in S}$. Some seats prioritize a certain type of individuals over others. The function $\sigma : S \rightarrow T \cup \{o\}$ specifies the type prioritized by each seat $s \in S$. If $\sigma(s) = t$ for some $t \in T$, then all type t individuals are ranked over all other individuals under π^s . That is, if $\sigma(s) = t$, for every $i \in I_t$ and $j \in I \setminus I_t$, we have $i \pi^s j$. If $\sigma(s) = o$, then we say s is an **open seat**, and it does not prioritize a certain type of individual. If $\sigma(s) = \sigma(s')$, then $\pi^s = \pi^{s'}$. That is, seats prioritizing the same type of individuals rank them in the same order. Let π_t and π_o be the priority

⁶In what follows, we say type t individuals instead of individuals with type t .

⁷Related types can also be understood in terms of graph connectivity. Define a type-graph as the graph whose vertex set is the set of types T and create an edge between each pair of types if and only if $I_t \cap I_{t'} \neq \emptyset$. Two types are related if and only if they are both contained within the same connected component of the type-graph.

orders of seats ranking type t individuals and open seats, respectively. For each type $t \in T$, let $S_t = \{s \in S : \sigma(s) = t\}$ be the set of **priority seats for type t** , and $S_o = \{s \in S : \sigma(s) = o\}$ be the set of **open seats**. We say that seat s is a **type t seat** if $s \in S_t$, and an **open seat** if $s \in S_o$. For each type t' , let $S_{t'}^D = \cup_{t \in D_{t'}} S_t$ and $S_{-t'}^D = S \setminus S_{t'}^D$.

A **problem** is a tuple $(I, S, T, \sigma, \tau, \pi)$. In the rest of the paper, we represent a problem with I whenever it is convenient. A **matching** $\mu : I \rightarrow S \cup \{\emptyset\}$ is a function that assigns each individual to at most one seat such that $|\mu^{-1}(s)| = 1$. If $\mu(i) = \emptyset$, then individual i is unassigned. Let \mathcal{M} be the set of matchings. For any matching μ , let $|\mu| = |\{i \in I : \mu(i) \in S\}|$ be the number of individuals who are assigned to a seat.

We define some desirable properties for our setting. A matching μ is **non-wasteful** if, for every $i \in I$, $\mu(i) = \emptyset$ implies $|\mu| = |S|$. An individual i **justifiably envies** another individual i' under matching μ if $\mu(i') \in S$, $\mu(i) = \emptyset$, and $i \succ^{\pi(i')} i'$. A matching μ is **fair** if no individual justifiably envies another one under μ .

In our analysis, we have the following over-demanded assumption: the number of individuals of any type t is greater than the number of open seats and seats of types related to t .

Assumption 1 For each $t \in T$, $|I_t| > |S_o| + \sum_{t' \in R_t} |S_{t'}|$.

If $R_t = \{t\}$, then Assumption 1 reduces to $|I_t| > |S_o| + |S_t|$. Under Assumption 1, it is clear that all seats will be filled in any non-wasteful matching. Furthermore, in any fair matching, for every $t \in T$, the seats in S_t will be assigned to individuals in I_t . We formally prove these in the Appendix.

For a given subset of individuals $\bar{I} \subseteq I$, there may exist multiple fair and non-wasteful matchings. In particular, the order of the seats filled in a sequential way may determine which fair and non-wasteful matching is observed. A **precedence order** \triangleright is a linear

order over the seats in S . Given two seats $s, s' \in S$, let $s \triangleright s'$ mean that seat s is to be filled before seat s' . Let Δ be the set of all precedence orders. For a given subset of individuals $\bar{I} \subseteq I$ and precedence order \triangleright , the chosen set of individuals is denoted by $C(\triangleright, \bar{I})$ and we call $C(\cdot)$ a **choice function**. Implicitly, $C(\cdot)$ depends on the priority profile π , but since π is fixed through the analysis, we suppress it for brevity.

Let $s_k \triangleright s_{k+1}$ for all $k \in \{1, \dots, |S| - 1\}$. Then, $C(\triangleright, \bar{I})$ is determined as follows: Initially set $C(\triangleright, \bar{I}) = \emptyset$. Assign the individual with the highest priority under π^{s_1} among individuals in \bar{I} to s_1 , and add them to $C(\triangleright, \bar{I})$. Next, assign the individual with the highest priority under π^{s_2} among $\bar{I} \setminus C(\triangleright, \bar{I})$ to s_2 , and add them to $C(\triangleright, \bar{I})$. Continue until all seats are considered.

Given a subset of individuals \bar{I} and precedence order \triangleright , for each seat $s \in S$, $C^s(\triangleright, \bar{I})$ is the individual chosen for s by the choice function $C(\cdot)$. Let $C_t(\triangleright, \bar{I}) = C(\triangleright, \bar{I}) \cap \bar{I}_t$. Slightly abusing notation, for any $t \in T \cup \{o\}$, let $C^t(\triangleright, \bar{I}) = \cup_{s \in S_t} C^s(\triangleright, \bar{I})$ be the set of individuals assigned to seats in S_t .

We say that **matching μ is induced by precedence order \triangleright** if, for every seat $s \in S$, $\mu^{-1}(s) = C^s(\triangleright, I)$ and for every individual $i \notin C^s(\triangleright, I)$, $\mu(i) = \emptyset$. With a slight abuse of notation, if μ is induced by precedence order \triangleright , we use $\mu = C(\triangleright, I)$.

4 Results

We first investigate the relationship between the set of matchings induced by all possible precedence orders and the set of fair and non-wasteful matchings.

Proposition 1 *Under Assumption 1, the following hold:*

- (a) *If $\mu = C(\triangleright, I)$ for some $\triangleright \in \Delta$, then μ is fair and non-wasteful.*

(b) Suppose that for every $t \in T$ and $s \in S_t$, the relative ranking of individuals in I_t is the same as in π_o , i.e., $i \pi^s j$ if and only if $i \pi_o j$ for all $i, j \in I_t$. Then, for any fair and non-wasteful matching μ there exists a precedence order \triangleright such that $\mu = C(\triangleright, I)$.

Proof. We first prove Part (a). By Assumption 1 and the definition of choice function $C(\cdot)$, all seats are filled under μ , and therefore μ is non-wasteful. If $\mu(i) = s$, then Assumption 1 implies that $\sigma(s) \in \tau(i) \cup \{o\}$. In the calculation of $C(\triangleright, I)$ each seat s is filled with the individual with the highest π^s priority among the remaining individuals. That is, if $\mu(i) = \emptyset$ and $\mu(i') = s$ for some s , then $i' \pi^s i$, and μ is fair.

Next we prove Part (b). By Assumption 1, fairness, and non-wastefulness of μ , for every $t \in T$, every seat $s \in S_t$ is assigned to some individual in I_t and every open seat is filled. Let i_k be the individual who has the k^{th} highest priority under π_o . We consider individuals one by one according to π_o and construct a precedence order \triangleright such that $\mu = C(\triangleright, I)$. We start with i_1 . If $\mu(i_1) = \emptyset$, then by fairness of μ , $|S_m| = 0$ for every $m \in \tau(i_1) \cup \{o\}$. Otherwise, let $\mu(i_1)$ have the highest precedence order under \triangleright . By our construction, i_1 has the highest priority for $\mu(i_1)$, and $C(\triangleright, I)$ assigns i_1 to $\mu(i_1)$. Similarly, if $\mu(i_2) = \emptyset$, then by fairness of μ , $|S_m \setminus \{\mu(i_1)\}| = 0$ for every $m \in \tau(i_2) \cup \{o\}$. Otherwise, let $\mu(i_2)$ have the highest precedence under \triangleright among the seats in $S \setminus \{\mu(i_1)\}$. Individual i_2 has the highest priority for $\mu(i_2)$ among all individuals in $I \setminus \{i_1\}$, so $C(\triangleright, I)$ assigns i_2 to $\mu(i_2)$. Repeating these arguments for all the individuals one by one according to π_o , we construct a well-defined precedence order that induces the matching μ . ■

We would like to emphasize that if our conditions in Proposition 1 Part (b) do not hold, then some fair and non-wasteful matchings cannot be induced by some precedence order. We illustrate this situation in the following example.

Example 1 Let $I = \{i_1, i_2, i_3\}$, $T = \{t\}$, $\tau(i_1) = \tau(i_2) = \tau(i_3) = t$, $S = \{s_1, s_2\}$, $\sigma(s_1) = o$, $\sigma(s_2) = t$, $i_1 \pi_o i_2 \pi_o i_3$ and $i_2 \pi^{s_2} i_1 \pi^{s_2} i_3$. Then, the following matching is fair and non-wasteful: $\mu(i_1) = s_2$ and $\mu(i_2) = s_1$. However, for any $\triangleright \in \Delta$, $C(\triangleright, I)$ assigns i_1 to s_1 and i_2 to s_2 .

Proposition 1 implies that $C(\triangleright, I)$ is fair and non-wasteful for every $\triangleright \in \Delta$. Then, one can find the precedence order which maximizes the number of chosen type t^* individuals among the fair and non-wasteful matchings induced by a precedence order by computing $C(\triangleright, I)$ for all $\triangleright \in \Delta$. However, since we have $|\Delta| = |S|!$ precedence orders, calculating all such fair and non-wasteful matchings via enumeration is impractical. Providing a “systematic recipe”, which is independent of priority orders, for the construction of a precedence order to achieve maximum representation of targeted type individuals may be desirable to the market designer.⁸ In the general model, such a systematic recipe does not exist. In the following example, we illustrate this situation by showing that the maximum representation of targeted type individuals is achieved under two different precedence orders depending on the priority order of open seats.⁹¹⁰

Example 2 Let $T = \{t_1, t_2, t^*\}$, $S_{t_1} = \{s_1\}$, $S_{t_2} = \{s_2\}$, $S_{t^*} = \emptyset$, $S_o = \{o_1, o_2\}$, $I_{t_1} = \{j_1, j_2, j_3\}$, $I_{t_2} = \{i_1, i_2, k_1, k_2\}$, and $I_{t^*} = \{k_1, k_2, k_3, k_4\}$. Consider precedence orders \triangleright and \triangleright' such that $o_1 \triangleright s_1 \triangleright s_2 \triangleright o_2$ and $s_1 \triangleright' o_1 \triangleright' s_2 \triangleright' o_2$. The priority orders of priority seats for type t_1 and t_2 are: $\pi_{t_1} : j_1 - j_2 - j_3 - \dots$, and $\pi_{t_2} : i_1 - i_2 - k_1 - k_2 - \dots$

We consider two priority orders for the open seats.

Case 1: $\pi_o = j_1 - i_1 - j_2 - k_1 - \dots$. Then, $C(\triangleright, I) = \{j_1, j_2, i_1, k_1\}$ and $C(\triangleright', I) = \{j_1, i_1, i_2, j_2\}$. The maximum number of type t^* individuals chosen in a fair and non-

⁸Formally, by systematic recipe we mean that given (I, S, T, σ, τ) and a type t , devise a method to construct \triangleright such that for each priority order π , representation of t is maximized.

⁹Having two different priority orders can be interpreted as having two different sets of individuals with different characteristics.

¹⁰Example 3 also shows a similar situation.

wasteful matching is one. In this case, \triangleright achieves the maximum representation for type t^* individuals but \triangleright' fails.

Case 2: $\pi_o = j_1 - k_1 - k_2 - i_1 - \dots$. Then, $C(\triangleright, I) = \{j_1, j_2, i_1, k_1\}$ and $C(\triangleright', I) = \{j_1, i_1, k_1, k_2\}$. The maximum number of type t^* individuals chosen in a fair and non-wasteful matching is two. In this case, \triangleright' achieves the maximum representation for type t^* individuals but \triangleright fails.

Although Example 2 shows that it is not easy to provide a systematic recipe to achieve maximum representation of a targeted type, we study whether it is possible to increase the representation of a targeted type for a given precedence order. We start our analysis by showing that we can increase the number of chosen type t^* individuals by processing the seats in $S_{t^*}^D$ after all other seats while preserving the relative order of the seats in $S_{-t^*}^D$.¹¹

Proposition 2 *Let I be the set of individuals and $\triangleright, \triangleright' \in \Delta$ such that*

- $s \triangleright s'$ if and only if $s \triangleright' s'$ for all $s, s' \in S_{t^*}^D$ or $s, s' \in S_{-t^*}^D$; and
- $s \triangleright' s'$ for all $s \in S_{-t^*}^D$ and $s' \in S_{t^*}^D$.

Under Assumption 1, the number of type t^ individuals selected under \triangleright' is weakly more than the one under \triangleright , i.e., $|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\triangleright', I)|$.*

Proof. If all seats in $S_{t^*}^D$ succeed all seats in $S_{-t^*}^D$ under \triangleright , then $\triangleright = \triangleright'$ and $C_{t^*}(\triangleright, I) = C_{t^*}(\triangleright', I)$. Suppose at least one seat $s^* \in S_{t^*}^D$ precedes at least one seat $\bar{s} \in S_{-t^*}^D$ under \triangleright .

¹¹Pathak et al. (2021) show that when we have a precedence order in which all seats with the same type are adjacent, we can increase the number of chosen type t individuals by swapping the precedence order of seats in S_t with the succeeding type t' seats. Pathak et al. (2021) show that when we have a precedence order in which all seats with the same type are adjacent, the earlier type t seats are processed the more selective it becomes (see Proposition 2). Although their result does not directly imply Proposition 2, we can iterate their proof idea to show a similar result.

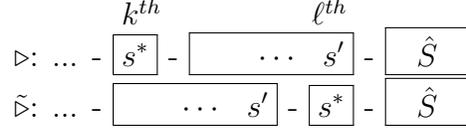


Figure 1: Illustration of \triangleright and $\tilde{\triangleright}$.

In our proof, at each step we will focus on an intermediate case in which we move one seat in $S_{t^*}^D$ after all seats in $S_{-t^*}^D$ one at a time while preserving the relative precedence order of other seats and show that such movement weakly increases the assignment of type t^* individuals. Such an increase will be observed when we repeat this movement one seat at a time for the other seats in $S_{t^*}^D$.

Let \hat{S} be the strict subset of seats in $S_{t^*}^D$ with lower precedence under \triangleright than all other types of seats; that is, for each $\hat{s} \in \hat{S} \subsetneq S_{t^*}^D$, and $s \notin S \setminus \hat{S}$, we have $s \triangleright \hat{s}$. Let s^* be the the lowest precedence seat in $S_{t^*}^D \setminus \hat{S}$ under \triangleright , and s' be the lowest precedence seat in $S_{-t^*}^D$ under \triangleright . We denote $\tilde{\triangleright}$ as the precedence order obtained from \triangleright by moving seat s^* just after s' , while keeping the relative order of all other seats the same. Suppose s^* is the k^{th} and ℓ^{th} seat under precedence orders \triangleright and $\tilde{\triangleright}$, respectively (See Figure 1). By construction, $|S| \geq \ell > k$. The m^{th} seat under \triangleright is the same seat as the m^{th} seat under $\tilde{\triangleright}$ where $m < k$ or $m > \ell$. Moreover, the m^{th} seat under $\tilde{\triangleright}$ is the same seat as the $(m+1)^{th}$ seat under \triangleright where $\ell > m \geq k$.

We denote the set of individuals selected by choice function $C(\cdot)$ up until the m^{th} slot under \triangleright and $\tilde{\triangleright}$ with $C^{1:m}(\triangleright, I)$ and $C^{1:m}(\tilde{\triangleright}, I)$, respectively. Let $B^m = I \setminus C^{1:m-1}(\triangleright, I)$ and $\tilde{B}^m = I \setminus C^{1:m-1}(\tilde{\triangleright}, I)$. That is, the set of individuals considered for the m^{th} seat under $C(\tilde{\triangleright}, I)$ and $C(\triangleright, I)$ are \tilde{B}^m and B^m , respectively. We denote the individual assigned to the m^{th} seat under $C(\triangleright, I)$ with i_m .

For $m < k$, since the m^{th} seat under \triangleright and $\tilde{\triangleright}$ are the same, $C^{1:m}(\triangleright, I) = C^{1:m}(\tilde{\triangleright}, I)$. We claim that for $\ell > m \geq k$, $C^{1:m}(\tilde{\triangleright}, I) \subset C^{1:m+1}(\triangleright, I)$ and $|C^{1:m+1}(\triangleright, I) \setminus C^{1:m}(\tilde{\triangleright}, I)| = 1$. If

the former part holds, then the latter part of the claim directly follows from Assumption 1, i.e., all seats are filled under both choice functions. We prove the former part of the claim via induction. We start with $m = k$. By our construction, $\tilde{B}^k \supset B^{k+1}$ and $\{i_k\} = \tilde{B}^k \setminus B^{k+1}$. Recall that the types of seat $k + 1$ under \triangleright and seat k under $\tilde{\triangleright}$ are the same. Hence, the individual selected for k^{th} seat under $C(\tilde{\triangleright}, I)$ is either i_k or i_{k+1} . Then, $C^{1:k}(\tilde{\triangleright}, I) \subset C^{1:k+1}(\triangleright, I)$ and $C^{1:k+1}(\triangleright, I) \setminus C^{1:k}(\tilde{\triangleright}, I)$ is either $\{i_k\}$ or $\{i_{k+1}\}$.

Suppose our claim holds for all m such that $\ell > \bar{m} > m \geq k$. That is, $C^{1:\bar{m}-1}(\tilde{\triangleright}, I) \subset C^{1:\bar{m}}(\triangleright, I)$ and $|C^{1:\bar{m}}(\triangleright, I) \setminus C^{1:\bar{m}-1}(\tilde{\triangleright}, I)| = 1$. Let $C^{1:\bar{m}}(\triangleright, I) \setminus C^{1:\bar{m}-1}(\tilde{\triangleright}, I) = \{j\}$. Then, $\tilde{B}^{\bar{m}} \supset B^{\bar{m}+1}$ and $\{j\} = \tilde{B}^{\bar{m}} \setminus B^{\bar{m}+1}$. Since the $(\bar{m} + 1)^{\text{th}}$ seat under \triangleright and the \bar{m}^{th} seat under $\tilde{\triangleright}$ are the same, so they have the same type. Hence, the individual selected for \bar{m}^{th} seat under $C(\tilde{\triangleright}, I)$ is either j or $i_{\bar{m}+1}$. Then, $C^{1:\bar{m}}(\tilde{\triangleright}, I) \subset C^{1:\bar{m}+1}(\triangleright, I)$ and $C^{1:\bar{m}+1}(\triangleright, I) \setminus C^{1:\bar{m}}(\tilde{\triangleright}, I)$ is either $\{j\}$ or $\{i_{\bar{m}+1}\}$. Then, $|C_{t^*}^{1:\ell-1}(\tilde{\triangleright}, I)| \geq |C_{t^*}^{1:\ell}(\triangleright, I)| - 1$. Since the ℓ^{th} seat under $\tilde{\triangleright}$ is in $S_{t^*}^D$, the chosen individual is in I_{t^*} . Then, we have $|C_{t^*}^{1:\ell}(\tilde{\triangleright}, I)| \geq |C_{t^*}^{1:\ell}(\triangleright, I)|$. If $\ell = |S|$, then we are done. If $\ell < |S|$, the last $|S| - \ell$ seats, which are all in $S_{t^*}^D$, will be filled by individuals in I_{t^*} . Hence, $|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\tilde{\triangleright}, I)|$.

If $\tilde{\triangleright} = \triangleright'$, then we are done. Otherwise, there exists another seat in $S_{t^*}^D$, say \hat{s} , which precedes some other seat in $S_{-t^*}^D$, possibly an open seat, under $\tilde{\triangleright}$. We repeat the process above by replacing s^* with \hat{s} , and \triangleright with $\tilde{\triangleright}$. Due to the finite number of seats, we ultimately achieve \triangleright' after finite repetition of this process and in each repetition the type t^* assignment weakly increases. Thus, we have $|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\tilde{\triangleright}, I)| \leq \dots \leq |C_{t^*}(\triangleright', I)|$. ■

We would like to remark that under precedence order \triangleright' , moving all type t^* seats after all other seats in $S_{t^*}^D$ cannot increase the number of assigned type t^* individuals further. By Assumption 1 and the definition of D_{t^*} , all seats in $S_{t^*}^D$ are filled by type t^* individuals. Hence, reordering the seats in $S_{t^*}^D$ under \triangleright' does not change the number of

assigned type t^* individuals.

One might consider that filling unrelated seats of type t^* before all other seats will result in an increase in the number of selected type t^* individuals. That is, given a precedence order \triangleright , suppose that we move all seats that are unrelated to type t^* in front of all other seats while preserving the relative precedence order of the related and unrelated seats within each group. Does this movement increase the number of selected type t^* individuals compared to \triangleright ? Case 1 in Example 2 shows that this is not always true. That is, we do not have a systematic procedure that will further increase the number of the selected type t^* individuals for any problem given a precedence order without enforcing any restrictions. We therefore analyze restrictions on precedence orders and the types of individuals considered to further maximize the presence of targeted type individuals.

We will show that whenever all open seats are adjacent under some precedence order, we can further increase the number of selected type t^* individuals. We do this by moving all seats of unrelated types of t^* just in front of all open seats without changing the relative precedence, in addition to the movement prescribed in Proposition 2.

Proposition 3 *Let \triangleright be a precedence order in which all open seats are adjacent to each other, i.e., there does not exist $s, s' \in S_o$ and $s'' \notin S_o$ such that $s \triangleright s'' \triangleright s'$. Let I be the set of individuals and \triangleright' be a precedence order obtained from \triangleright by moving*

- *all seats in $S_{t^*}^D$ after all other seats, and*
- *all seats of types unrelated to type t^* that are preceded by open seats under \triangleright just before all open seats*

while keeping the relative order of all other seats the same. Then, under Assumption 1,

$$|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\triangleright', I)|.$$

Proof. Suppose $\triangleright \neq \triangleright'$. We will prove the desired result by obtaining \triangleright' through a series of transformations to \triangleright .

Let $\hat{\triangleright}$ be a precedence order obtained from \triangleright by moving all seats in $S_{t^*}^D$ after all other seats while preserving the relative order of seats within $S_{-t^*}^D$ and $S_{t^*}^D$. By Proposition 2, $|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\hat{\triangleright}, I)|$.

Let $\bar{\triangleright}$ be obtained from $\hat{\triangleright}$ by rearranging the seats that are between the open seats and the seats in $S_{t^*}^D$ as follows: all seats unrelated to type t^* now precede all seats related to type t^* and the relative precedence order within each group (unrelated/related) is preserved. Since, under Assumption 1, no individual of type t can fill a seat with a type that is unrelated to type t under any precedence order, and the relative order within each group is preserved when we obtain $\bar{\triangleright}$ from $\hat{\triangleright}$, we have $C(\bar{\triangleright}, I) = C(\hat{\triangleright}, I)$.

Notice that we can obtain \triangleright' from $\bar{\triangleright}$ by moving all seats unrelated to t^* just in front of all open seats, while preserving the relative precedence order of all of the other seats. Under $\bar{\triangleright}$, suppose the first open seat is the k^{th} seat. Since we do not change the order of seats preceding the k^{th} seat when constructing \triangleright' from $\bar{\triangleright}$, $C^{1:k-1}(\bar{\triangleright}, I) = C^{1:k-1}(\triangleright', I)$. We now compare the assignment to the remaining seats under $\bar{\triangleright}$ and \triangleright' . Recall that under $\bar{\triangleright}$ and \triangleright' the order of the seats after the initial identical block, i.e., the first $k - 1$ seats, are illustrated in Figure 2.

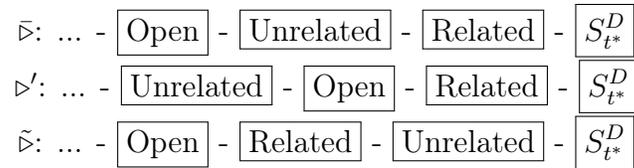


Figure 2: Illustration of $\bar{\triangleright}$, \triangleright' and $\tilde{\triangleright}$.

We first consider precedence order $\bar{\triangleright}$. As explained above, the set of individuals who are prioritized for seats related to t^* and seats unrelated to t^* are disjoint. Hence, under

Assumption 1, reordering the unrelated and related seats between the open seats and the seats in $S_{t^*}^D$ while keeping the relative order of the seats within each group does not change the set of selected individuals. That is, we obtain precedence order $\tilde{\triangleright}$ from $\bar{\triangleright}$ by moving the unrelated seats after the open seat block to just before the seats in $S_{t^*}^D$ (See Figure 2), then $C(\bar{\triangleright}, I) = C(\tilde{\triangleright}, I)$.

Consider $\tilde{\triangleright}$ and \triangleright' . First recall that type t^* individuals can be assigned to open and seats of related types. Since we assign some of the individuals that are type unrelated to t^* to the seats unrelated to t^* first under precedence order \triangleright' , the set of unassigned applicants left to be assigned seats in the open block under \triangleright' is a subset of the same under $\tilde{\triangleright}$. Thus, weakly more individuals of type in R_{t^*} receive an open seat under \triangleright' than under $\tilde{\triangleright}$. Finally, each individual who has a type in R_{t^*} and assigned under $\tilde{\triangleright}$ to some seat in the open or related block will be assigned under \triangleright' to some seat; this is a consequence of lowered competition due to the earlier assignment of unrelated type individuals.¹² Restricted to type t^* applicants, $|C_{t^*}(\tilde{\triangleright}, I)| \leq |C_{t^*}(\triangleright', I)|$. Hence, we have

$$|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\hat{\triangleright}, I)| = |C_{t^*}(\bar{\triangleright}, I)| = |C_{t^*}(\tilde{\triangleright}, I)| \leq |C_{t^*}(\triangleright', I)|.$$

■

If all types related to t^* are included by it, i.e., $R_{t^*} = D_{t^*}$, then Proposition 3 implies that among the precedence orders in which all open seats are adjacent, we can increase the number of type t^* assignment by moving all other seats in front of the open seats and all related seats after the open seats. We show that this result holds unconditionally on whether open seats are adjacent or not.

Proposition 4 *Suppose $R_{t^*} = D_{t^*}$. Let I be the set of individuals, and \triangleright and \triangleright' be*

¹²This is also referred to as the population monotonicity of the Deferred Acceptance mechanism.

precedence orders such that \triangleright' is obtained from \triangleright by moving

- all seats in $S_{t^*}^D$ after all other seats, and
- all seats of types unrelated to type t^* just before all open seats

while keeping the relative order of all seats within each S_o , $S_{t^*}^D$, and $S \setminus (S_o \cup S_{t^*}^D)$ the same. Under Assumption 1, $|C_{t^*}(\triangleright, I)| \leq |C_{t^*}(\triangleright', I)|$.

Proof. We first obtain precedence order $\bar{\triangleright}$ from \triangleright by moving all seats in $S_{t^*}^D$ after all other seats, while preserving the relative order of all other seats. By Proposition 2, $|C_{t^*}(\bar{\triangleright}, I)| \geq |C_{t^*}(\triangleright, I)|$. Under $\bar{\triangleright}$ all open seats precede seats in $S_{t^*}^D$. If all seats in $S \setminus (S_{t^*}^D \cup S_o)$ precede open seats under $\bar{\triangleright}$, then we are done. Otherwise, there exists at least one open seat preceding some seat in $S \setminus (S_{t^*}^D \cup S_o)$ under $\bar{\triangleright}$. If all open seats are adjacent under $\bar{\triangleright}$, then Proposition 3 implies that we can increase the number of type t^* individuals selected by $C(\cdot)$ by moving all unrelated type seats, i.e., seats in $S \setminus (S_{t^*}^D \cup S_o)$, in front of all open seats under $\bar{\triangleright}$.

Suppose that not all open seats are adjacent under $\bar{\triangleright}$. For the rest of the proof, we call the set of adjacent open seats as an open seat block. We obtain precedence order $\check{\triangleright}$ from $\bar{\triangleright}$ by moving all unrelated type seats between the last two open seat blocks just in front of the penultimate open seat block. Recall that there is no type t^* related seats between these two open seat blocks under $\bar{\triangleright}$. Since the precedence order of all other seats are the same under both $\bar{\triangleright}$ and $\check{\triangleright}$, it is sufficient to focus on the assignment of the last two open seat blocks, the seats between these open seat blocks, and type t^* related seats under $\bar{\triangleright}$ and corresponding seats under $\check{\triangleright}$. We denote the last two open seat blocks with S^{o1} and S^{o2} , seats between them under $\bar{\triangleright}$ with S^{-t^*} and type t^* related seats with S^{t^*} (See Figure 3).¹³ We denote the set of individuals assigned to these seats under $C(\bar{\triangleright}, I)$

¹³Notice that $S^{t^*} = S_{t^*}^D$.

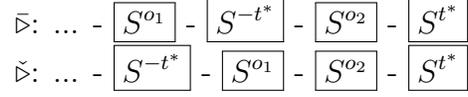


Figure 3: Illustration of $\bar{\triangleright}$ and $\check{\triangleright}$.

and $C(\check{\triangleright}, I)$ with $X^{o_1}, X^{o_2}, X^{-t^*}, X^{t^*}$, and $Y^{o_1}, Y^{o_2}, Y^{-t^*}, Y^{t^*}$, respectively. Under $C(\bar{\triangleright}, I)$, we denote the set of applicants waiting to be assigned to S^{o_1}, S^{-t^*} and S^{o_2} with M_{o_1}, M_{-t^*} and M_{o_2} , respectively. Under $C(\check{\triangleright}, I)$, we denote the set of applicants waiting to be assigned to S^{o_1}, S^{-t^*} and S^{o_2} with N_{o_1}, N_{-t^*} and N_{o_2} , respectively. Notice that $M_{o_1} = N_{-t^*}$. By the definition of the choice functions, we have $X^{o_1} \subseteq Y^{-t^*} \cup Y^{o_1}$ and $Y^{-t^*} \subseteq X^{o_1} \cup X^{-t^*}$. That is, any individual assigned to first open seat block, S^{o_1} , under $C(\bar{\triangleright}, I)$ is assigned to a seat in either S^{-t^*} or S^{o_1} under $C(\check{\triangleright}, I)$. Similarly, any individual assigned to S^{-t^*} under $C(\check{\triangleright}, I)$ is assigned to a seat in either S^{o_1} or S^{-t^*} under $C(\bar{\triangleright}, I)$. Then, $Y^{-t^*} \subseteq X^{o_1} \cup X^{-t^*}$ implies that $N_{-t^*} \supseteq N_{o_1} \supseteq M_{o_2}$.

If $Y^{-t^*} \cup Y^{o_1} = X^{-t^*} \cup X^{o_1}$, then we have $C(\bar{\triangleright}, I) = C(\check{\triangleright}, I)$. This follows from the fact that the precedence order of the remaining seats under $\bar{\triangleright}$ and $\check{\triangleright}$ is the same. Hence, the desired result follows.

Suppose $Y^{-t^*} \cup Y^{o_1} \neq X^{-t^*} \cup X^{o_1}$. Let $K = (X^{-t^*} \cup X^{o_1}) \setminus (Y^{-t^*} \cup Y^{o_1})$ and $L = (Y^{-t^*} \cup Y^{o_1}) \setminus (X^{-t^*} \cup X^{o_1})$. By Assumption 1, since all seats are filled under both $C(\bar{\triangleright}, I)$ and $C(\check{\triangleright}, I)$ we have $|K| = |L|$. Moreover, $K \subseteq X^{-t^*}$ and $L \subseteq Y^{o_1}$. Since $L \cap (X^{-t^*} \cup X^{o_1}) = \emptyset$ and $N_{-t^*} \supseteq N_{o_1} \supseteq M_{o_2}$, $L \subseteq M_{o_2} \subseteq N_{-t^*}$. By definition, $((Y^{-t^*} \cup Y^{o_1}) \setminus L) \cap M_{o_2} = \emptyset$. Hence, for any $\ell \in L$ and $i \in M_{o_2} \setminus L$ we have $\ell \pi_o i$. Then, we have the following two cases:

Case 1: $|L| \geq |S^{o_2}|$. In this case, if $i \in X^{o_2}$, then $i \in L$. That is, $L \supseteq X^{o_2}$. Then, $X^{o_1} \cup X^{o_2} \subseteq Y^{o_1} \cup Y^{o_2} \cup Y^{-t^*}$. Since $(X^{o_1} \cup X^{o_2} \cup X^{-t^*}) \cap I_{t^*} \subseteq X^{o_1} \cup X^{o_2}$ and all seats in S^{t^*} are filled with I_{t^*} individuals under $C(\check{\triangleright}, I)$, we have $|C_{t^*}(\check{\triangleright}, I)| \geq |C_{t^*}(\bar{\triangleright}, I)| \geq |C_{t^*}(\triangleright, I)|$.

Case 2: $|L| < |S^{o2}|$. Since $\ell \pi_o i$ for any $\ell \in L$ and $i \in M_{o2} \setminus L$, we have $L \subset X^{o2}$. Moreover, $|N_{o2} \setminus M_{o2}| = |M_{o2} \setminus N_{o2}| = |L| = |K|$. Then, any $i \in X^{o2} \setminus L$ is one of the $|S^{o2}|$ individuals with the highest priority in N_{o2} , i.e., $i \in Y^{o2}$. Then, $i \in (X^{o1} \cup X^{o2} \cup X^{-t^*}) \cap I_{t^*}$ implies that $i \in (Y^{o1} \cup Y^{o2} \cup Y^{-t^*})$. Since all seats in S^{t^*} are filled with I_{t^*} individuals under $C(\check{\triangleright}, I)$, we have $|C_{t^*}(\check{\triangleright}, I)| \geq |C_{t^*}(\bar{\triangleright}, I)| \geq |C_{t^*}(\triangleright, I)|$.

If $\check{\triangleright} = \triangleright'$, then we are done. Otherwise, there exist two open slot blocks under $\check{\triangleright}$ which are not adjacent to each other. Hence, we can repeat steps described above and attain \triangleright' and each repetition weakly increases the number of selected type t^* individuals.

■

So far, we have not discussed whether we can further increase the number of selected type t^* individuals by rearranging the precedence orders of unrelated type seats or related type seats. The next two examples show that we cannot prescribe a precedence order arrangement among either related or unrelated type seats to further increase the number of selected type t^* individuals, without priority information.

Example 3 Let $T = \{t_1, t_2, t^*\}$, $S_{t_1} = \{s_1\}$, $S_{t_2} = \{s_2\}$, $S_{t^*} = \emptyset$, $S_o = \{o\}$, $I_{t_1} = \{i, j_1\}$, $I_{t_2} = \{i, j_2\}$, and $I_{t^*} = \{k\}$. Then, $R_{t^*} = D_{t^*} = \{t^*\}$ and $U_{t^*} = \{t_1, t_2\}$. Consider precedence orders \triangleright and \triangleright' such that $s_1 \triangleright s_2 \triangleright o$ and $s_2 \triangleright' s_1 \triangleright' o$. Independent of the priority orders of the individuals, Proposition 4 implies that there does not exist any other precedence order which increases the number of selected type t^* individuals compared to both \triangleright and \triangleright' .

Let $\pi^{s_1} : i - j_1 - \dots$ and $\pi^{s_2} : i - j_2 - \dots$. We consider two possible cases over the priority order of the open seat such that \triangleright' and \triangleright maximize the number of selected type t^* individuals in the first and second cases, respectively.

Case 1: $\pi_o : i - j_1 - k - j_2$. Then, $C(\triangleright, I) = \{i, j_1, j_2\}$ and $C(\triangleright', I) = \{i, j_1, k\}$. The maximum number of type t^* individuals is selected under \triangleright' .

Case 2: $\pi_o : i - j_2 - k - j_1$. Then, $C(\triangleright, I) = \{i, j_2, k\}$ and $C(\triangleright', I) = \{i, j_1, j_2\}$.

The maximum number of type t^* individuals is selected under \triangleright .

Example 4 Let $T = \{t_1, t_2, t^*\}$, $S_{t_1} = \{s_1\}$, $S_{t_2} = \{s_2\}$, $S_{t^*} = \emptyset$, $S_o = \{o\}$, $I_{t_1} = \{i_1, i_2, k\}$, $I_{t_2} = \{i_1, j_1, k\}$, and $I_{t^*} = \{k\}$. Then, $R_{t^*} = \{t_1, t_2, t^*\}$ and $U_{t^*} = \emptyset$. Consider precedence orders \triangleright and \triangleright' such that $o \triangleright s_1 \triangleright s_2$ and $o \triangleright' s_2 \triangleright' s_1$.

Let $\pi^{s_1} : i_1 - i_2 - k \dots$ and $\pi^{s_2} : i_1 - j_1 - k \dots$. We consider two possible cases over the priority order of the open seat such that \triangleright and \triangleright' maximize the number of selected type t^* individuals in the first and second cases, respectively.

Case 1: $\pi_o : j_1 - i_1 - k - i_2$. Then, $C(\triangleright, I) = \{j_1, i_1, k\}$, $C(\triangleright', I) = \{j_1, i_1, i_2\}$, and $|C_{t^*}(\triangleright, I)| > |C_{t^*}(\triangleright', I)|$.

Case 2: $\pi_o : i_2 - i_1 - k - j_1$. Then, $C(\triangleright, I) = \{i_2, i_1, j_1\}$, $C(\triangleright', I) = \{i_2, i_1, k\}$, and $|C_{t^*}(\triangleright, I)| < |C_{t^*}(\triangleright', I)|$.

5 Conclusion

In this paper, we study the allocation of homogeneous objects to individuals. Each individual may belong to more than one type and some objects prioritize individuals with a specific type over others. In this environment we prescribe ways to systemically increase the number of selected individuals with a targeted type via sequential processing, while satisfying non-wastefulness and fairness criteria.

We show that sequential processing will always yield a fair and non-wasteful allocation, and under certain conditions on the priority orders, every fair and non-wasteful allocation may be achieved with sequential processing. We further show that there is no single way to implement sequential processing that will guarantee the maximum representation of targeted type individuals for all problems.

Following these general results for sequential processing techniques, we describe adjustments that can be made to a given precedence order to weakly increase the presence of targeted type individuals in the resulting allocation. These adjustments rely on the nature of the included, related, and unrelated types to the targeted type t^* —independent from priority information.

Following the conclusions of Examples 2, 3, and 4, further work analyzing how best to increase the presence of targeted groups may take advantage of the structure of priority orders. For some problems, a stronger characterization of the related types may extend the results discussed above. In particular, two types might be related to each other due to only one individual i . If i is accepted when the choice function is applied to a precedence order, then we may treat these two types as unrelated for subsequent seats. The following example illustrates this intuition.

Example 5 *Let $T = \{t, t^*\}$, $S_o = \{o_1, o_2\}$, $S_t = \{s_t\}$, $S_{t^*} = \{s_{t^*}\}$, $I_t = \{i, j, k\}$, and $I_{t^*} = \{i, m, n\}$. Let $\pi_o : i-j-m-n-k$, $\pi^{s_t} : i-j-k-m-n$, and $\pi^{s_{t^*}} : m-n-i-j-k$. Note that $R_{t^*} = T$.*

In order to increase type t^ representation, Proposition 2 says we should process s_{t^*} last, but our results give no further guidance. Consider precedence order \triangleright such that $o_1 \triangleright o_2 \triangleright s_t \triangleright s_{t^*}$. Then, we have $C(\triangleright, I) = \{i, j, k, m\}$. After i is assigned to object o_1 , types t and t^* are no longer related when considering only individuals in $I \setminus \{i\}$.*

Seeing that t is now unrelated to t^ , Proposition 3 suggests that processing s_t before o_2 weakly increases the number of selected type t^* individuals. Let \triangleright' be the precedence order obtained from \triangleright by moving s_t between two open objects, i.e., $o_1 \triangleright' s_t \triangleright' o_2 \triangleright' s_{t^*}$. Then, we have $C(\triangleright', I) = \{i, j, m, n\}$ and $|C_{t^*}(\triangleright, I)| < |C_{t^*}(\triangleright', I)|$.*

Adjusting precedence orders in a dynamic fashion, or some other form of dynamic sequential processing, may allow for a greater presence of targeted type individuals under

a greater variety of problems.

References

- AYGÜN, O. AND I. BÓ (2021): “College Admission with Multidimensional Privileges: The Brazilian Affirmative Action Case,” *American Economic Journal: Microeconomics* (Forthcoming).
- AYGÜN, O. AND B. TURHAN (2020a): “Direct Matching Mechanism Design for India with Comprehensive Affirmative Action,” Working paper.
- (2020b): “Dynamic Reserves in Matching Markets,” *Journal of Economic Theory*, 188, 105069.
- DELACRÉTAZ, D. (2021): “Processing Reserves Simultaneously,” in *Proceedings of the 22nd ACM Conference on Economics and Computation*, ACM.
- DUR, U., P. PATHAK, S. D. KOMINERS, AND T. SÖNMEZ (2018): “Reserve Design: Unintended Consequences and The Demise of Boston’s Walk Zones,” *Journal of Political Economy*, 126, 2457–2479.
- DUR, U., P. PATHAK, AND T. SÖNMEZ (2020): “Explicit vs. Statistical Preferential Treatment in Affirmative Action: Theory and Evidence from Chicago’s Exam Schools,” *Journal Of Economic Theory*, 187, 104996.
- FRAGIADAKIS, D. AND P. TROYAN (2017): “Improving matching under hard distributional constraints,” *Theoretical Economics*, 12, 863–908.
- HAFALIR, I. E., M. B. YENMEZ, AND M. A. YILDIRIM (2013): “Effective affirmative action in school choice,” *Theoretical Economics*, 8, 325–363.

KOMINERS, S. D. AND T. SÖNMEZ (2016): “Matching with slot–specific priorities: Theory,” *Theoretical Economics*, 11, 683–710.

PATHAK, P., T. SÖNMEZ, M. U. ÜNVER, AND M. B. YENMEZ (2021): “Fair Allocation of Vaccines, Ventilators and Antiviral Treatments: Leaving No Ethical Value Behind in Health Care Rationing,” Working Paper.

PATHAK, P. A., A. REES-JONES, AND T. SÖNMEZ (2020): “Immigration Lottery Design: Engineered and Coincidental Consequences of H-1B Reforms,” Working Paper.

——— (2022): “Reversing Reserves,” Forthcoming in *Management Science*.

SÖNMEZ, T. AND M. B. YENMEZ (2019): “Affirmative Action in India via Vertical and Horizontal Reservations,” Working paper.

——— (2020): “Affirmative Action with Overlapping Reserves,” Working Paper.

——— (2021): “Affirmative Action in India via Vertical, Horizontal, and Overlapping Reservations,” Working paper.

Appendix A

Claim 1 *Under Assumption 1, if μ is non-wasteful, then all seats are filled.*

Proof. We will show that Assumption 1 implies $|I| > |S|$. In any matching where a seat is unfilled, $|I| > |S|$ implies that there is at least one agent that is unassigned—so μ violates non-wastefulness. We thus prove the claim by contraposition.

Observe that for each $t, t' \in T$, $t' \in R_t \Leftrightarrow t \in R_{t'}$. That is, the set of types that are related form connected components in the graph with types as vertices and edges as relatedness.

Let $\{T^k\}_{k \in K}$ be a partition of T comprised of related cells, where

- for each $k \in K$, and each $t, t' \in T^k$, $t' \in R_t$, and
- for each $t \in T^k$, and each $t' \notin T^k$, $t' \notin R_t$.

Consider $k \in K$. By Assumption 1, for each $t \in T^k$, $|I_t| > |S_o| + \sum_{t' \in R_t} |S_{t'}|$. Summing over all $t \in T^k$,

$$\begin{aligned} \sum_{t \in T^k} |I_t| &> |T^k| |S_o| + \sum_{t \in T^k} \sum_{t' \in R_t} |S_{t'}| \\ &= |T^k| |S_o| + |T^k| \sum_{t' \in T^k} |S_{t'}| \end{aligned}$$

where the equality follows from fact that for each $t, t' \in T$, $R_t = R_{t'}$. Thus,

$$\sum_{t \in T^k} |I_t| > |T^k| \left(|S_o| + \sum_{t' \in T^k} |S_{t'}| \right).$$

Since

$$|T^k| \left| \bigcup_{t \in T^k} I_t \right| \geq \sum_{t \in T^k} |I_t|,$$

combining with above, we have

$$\left| \bigcup_{t \in T^k} I_t \right| > |S_o| + \sum_{t' \in T^k} |S_{t'}|.$$

Summing over all $k \in K$,

$$\sum_{k \in K} \left| \bigcup_{t \in T^k} I_t \right| > \sum_{k \in K} \left(|S_o| + \sum_{t' \in T^k} |S_{t'}| \right).$$

Since K is a partition, the LHS equals $|I|$, and we have

$$\begin{aligned} |I| &> \sum_{k \in K} \left(|S_o| + \sum_{t' \in T^k} |S_{t'}| \right) \\ &= |K| |S_o| + \sum_{k \in K} \sum_{t' \in T^k} |S_{t'}| \\ &= |K| |S_o| + \sum_{t' \in T} |S_{t'}| \\ &\geq |S_o| + \sum_{t' \in T} |S_{t'}| \\ &= |S|. \end{aligned}$$

■

Claim 2 *Let Assumption 1 be true, and μ be a non-wasteful and fair matching. Then, all seats in S_t are assigned to individuals in I_t .*

Proof. In the proof of Claim 1, we show that Assumption 1 implies that $|I| > |S|$. Hence, in any matching at least one individual is unassigned. Let $\mu(i_1) = \emptyset$. Since μ is non-wasteful, Claim 1 implies that all seats are filled. Recall that if $s \in S_t$, then for any $i \in I_t$, and any $i' \in I \setminus I_t$, we have that $i \pi^s i'$. Then, for any $t \in \tau(i_1)$, all seats in S_t are assigned to individuals in I_t . Otherwise, μ cannot be fair.

Let I^1 be the set of individuals assigned to seats reserved for types in $\tau(i_1)$ and all individuals whose all types are included in $\tau(i_1)$. We obtain a reduced problem by removing all individuals in I^1 and their assignments under μ . Then, it is easy to see that

Assumption 1 holds when we consider the remaining individuals and seats. To see that, consider $t \in T \setminus \tau(i_1)$. By Assumption 1 in the original problem, $|I_t| > |S_o| + \sum_{t' \in R_t} |S_{t'}|$. In the reduced problem, we remove all agents in I_t that are assigned (they could have been a type in $\tau(i_1)$), and so the set of remaining agents in I_t is $I_t \setminus \{i \in I_t : \mu(i) \in \bigcup_{t' \in \tau(i_1)} S_{t'}\}$. By feasibility, for each $t' \in \tau(i_1)$, $|\{i \in I_t : \mu(i) \in S_{t'}\}| \leq |S_{t'}|$. Thus, we have

$$\left| I_t \setminus \left\{ i \in I_t : \mu(i) \in \bigcup_{t' \in \tau(i_1)} S_{t'} \right\} \right| > |S_o| + \sum_{t' \in R_t \setminus \tau(i_1)} |S_{t'}|,$$

which is the analog of Assumption 1 for the reduced problem.

Hence, there exists at least one individual $i_2 \in I \setminus I^1$ such that $\mu(i_2) = \emptyset$. Then, the same arguments above imply that all seats in S_t are assigned to individuals in I_t for any $t \in \tau(i_2)$. By repeating the arguments above, we can obtain the desired result. ■