Partitionable Choice Functions and Stability*

Umut Dur† Thayer Morrill‡ William Phan§

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Abstract

We consider the two-sided many-to-one matching problem and introduce a class of preferences reflecting natural forms of complementarities. For example, academic departments hire seniors then supporting juniors, teams recruit different roles and specialties starting with the critical ones, and firms hire workers at various levels starting with the executives. The key feature is that a firm can partition workers into types and prioritizes certain types before others. Despite this partitionability requirement of choice functions being weaker than substitutes—an essential condition concerning the existence of a stable assignment—we show that it still guarantees the existence of a stable assignment and is further a maximal domain for such.

Keywords: Matching Theory, Market Design, Complementarities

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†Department of Economics, North Carolina State University. Contact: udur@ncsu.edu.

‡Department of Economics, North Carolina State University. Contact: tsmorril@ncsu.edu.

§Department of Economics, North Carolina State University. Contact: wphan@ncsu.edu.

1 Introduction

We provide a new intuitive class of preferences featuring complementarities that guarantees a stable assignment in the two-sided many-to-one matching problem. The preferences reflect rather natural occurrences. For example, firms take greater care in hiring their CEO and executive staff than associates and analysts; an economics department recruits seniors then juniors then finally graduate students; a football team prioritizes the quarterback then other positions. While preferences such as these may not be substitutable—which would guarantee existence of a stable outcome—they are aligned in such a way where we can divide the problem into a sequence of smaller ones where solutions do exist.

More precisely, consider the matching problem in the context of the medical labor market. A set of hospitals wishes to hire from a set of doctors and nurses. Each doctor and each nurse has a preference over which hospital to join. Hospitals' choices over applicants are as follows: For each set of applicants, they choose their favorite set of doctors first, then choose nurses based on the specialty of the chosen doctors and the aim of having enough nurses to support the doctors.[1] Note that the two are complements (a particular nurse is chosen only if a particular doctor is chosen), and so hospitals' choice functions violate the substitutes condition.

For this problem, a stable assignment always exists. If we ignore the nurses, then there is a stable assignment of doctors to hospitals. Simply run the Deferred Acceptance mechanism (DA) with just these two groups. Fixing this assignment, update the hospitals rankings over nurses to reflect possible changes based on the doctors received and run DA again to assign nurses. No doctor can be a part of a coalition that blocks this assignment, since the assignment of doctors/hospitals is stable. Similarly, no doctor and nurse pair can block together, since no hospital wishes to select an alternative doctor. Finally, no nurse can block: if they are in a different specialty than the chosen doctor, then the hospital is not interested, and if they are in the same, then the hospital already has a preferred nurse.

[1]This example is reminiscent of (Danilov, 2003); however, in our environment, hospitals have preferences over arbitrary subsets of all types of agents, whereas they consider matchings that are tuples consisting of only one agent of every type.

With this scenario in mind, we introduce the concept of a *partitionable* choice function embodying the above intuition. If hospitals can collectively partition the medical staff into distinct groups and choices are aligned in a particular manner, then we show that it is not necessary that the staff are substitutes across these groups. Given an ordered partition of staff π, we define a choice function to be π-partitionable if (i) staff are substitutes within a group, and (ii) the choice of staff in one group does not depend on the staff available in a later/downstream group. A profile of choice functions is partitionable if there exists an ordered partition π such that each choice function is π-partitionable.[2] Note that any substitutable choice function can be partitioned by the trivial one consisting of all agents in one group; hence, partitionability is a weaker requirement than substitutability.

We prove that there always exists a stable assignment when choice functions are partitionable (Theorem 1). Although in our example in the introduction, doctor and nurse "types" are given exogenously, the statement holds for arbitrary profiles of choice functions where staff *can* be partitioned into such types.[3] We define the Sequential Deferred Acceptance (SDA) mechanism with respect to an order on staff types, and show that it results in a stable assignment.[4] Crucially, the order in which we run SDA aligns with that of the ordered partition π "rationalizing" the profile of choice functions. We also show a maximal domain result: If there is even one agent whose choice function is non-partitionable (while the others' are partitionable), then stable assignments are not guaranteed to exist (Proposition 2).

Finally, we provide an additional application of our results even when Theorem 1 cannot be directly invoked. Consider the following scenario: Each hospital wishes to hire a doctor only if it can support her with a specified number of nurses and vice versa. Despite the presence of pure complements, certain subdomains of problems (mainly dependent on the number of staff within each group) allow for a stable and strategy-proof mechanism which we refer to

[2]Implicitly, we consider the hospitals' choice functions collectively instead of individually, and our theoretical exercise is to identify a property of a profile of choice functions that is sufficient to guarantee the existence of a stable assignment. We may view the substitutes domain as a rectangular domain for profiles of choice functions, and the partitionable domain, a non-rectangular (π must be consistent across agents) superset of the substitutes domain.

[3]A similar exercise is performed with single-peaked preferences: Can the set of alternatives be ordered so that each agent's preference is single-peaked (with respect to this order)?

[4]When all agents have the same type, i.e., the partition is trivial, the SDA is equivalent to DA.

as the SDA with Placeholders (Theorem 2). More generally, we observe these preferences in labor markets where completion of a task requires a certain number of employees with different specialties or roles. Finally, diversity policies which require exact representation of subgroups within teams, schools, and colleges are examples of such pure complements.

The outline of the paper is as follows: In Section 2, we discuss the related literature. In Section 3, we formally define the model. In Section 3.1, we propose and discuss partitionability. In Section 3.2 we define the SDA mechanism, and establish the existence and maximal domain results. In Section 4, we discuss an extension of the SDA applied to an environment where agents have pure complementarities. Finally, we conclude in Section 5.

2 Related Literature

This paper considers a two-sided matching problem composed of agents with preferences (or choice functions) over the agents on the other side. Some examples of such problems are college admission programs, labor markets, hospital-residency matching programs, and student assignment systems (Gale and Shapley, 1962; Kelso and Crawford, 1982; Roth, 1982; Roth and Sotomayor, 1992; Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). For the survival and success of these matching markets, obtaining a stable matching as a final outcome is identified as critical (Roth, 1984, 1991; Roth and Sotomayor, 1992).

Existence of a stable outcome can be guaranteed when the multi-unit demand sides' choices over the agents on the other side of the market satisfy certain conditions. If the choice function of each agent on the multi-unit demand side satisfies the substitutes condition,[5] then existence of stable matching is guaranteed in any problem (Kelso and Crawford, 1982; Roth and Sotomayor, 1992; Hatfield and Milgrom, 2005; Hatfield and Kojima, 2010). Moreover, when only one contract can be signed between two agents, then the substitutes condition can be considered a necessary condition (Hatfield and Kojima, 2008). The substitutes domain

[5] Some variants of substitutability in the matching with contracts literature, such as bilateral substitutability, are equivalent to substitutability when there is only a single contract available for each potential match.

is thus a maximal domain for stability.

In our analysis of the standard two-sided matching problem, we introduce a condition—partitionability—that is weaker than substitutability, yet still guarantees the existence of a stable matching. This result does not contradict Hatfield and Kojima (2008) as they consider a related but different exercise. Hatfield and Kojima (2008) show that so long as one hospital's choice function violates substitutes, even if all other hospitals have capacity for only one doctor, then a stable assignment need not exist.[6,7,8] However, there is at least one reason why this result is not as general as it might first appear. Implicit in their argument is the assumption of *unrestricted domain*. Specifically, for the result to hold the "many" side of the market, hospitals, must be able to rank doctors in any possible order. The literature has overlooked this condition since in many environments it is innocuous; however, in others, such as choosing students from many grades, it may not be particularly appropriate. If a school is choosing for one grade, it would only rank students of that grade; it would never be the case that students of different grades would be interspersed among its rankings. Similarly, unrestricted domain requires that it is possible that each hospital could rank doctors in any order. This would mean the hospital could rank a doctor with specialty A ahead of a doctor with specialty B ahead of another doctor with specialty A, regardless of what these specialties are. However, there is a large number of specialties and hospitals only offer a limited number of services. We doubt any individual hospital offers every conceivable specialty, so it is unreasonable to assume that each hospital might have any conceivable ordering of doctors.

[6] Rostek and Yoder (2020) conduct a similar exercise but with *complementary* preferences. Their domain and the substitutes domain are unrelated (have empty intersection), while our partitionability condition includes the latter as a special case.

[7] Che et al. (2019) considers a large market matching model, and shows that approximately stable matchings exist without the substitutes condition.

[8] Hatfield et al. (2019) show that in the matching with contracts framework the cumulative offer process is the unique strategy-proof and stable mechanism when three conditions on multi-unit demand side choice function are satisfied—observable substitutes, observable size monotonicity, and non-manipulability via contractual terms. Moreover, if the choice function of a firm does not satisfy any of these three conditions, then there exist instances in which all other firms have unit demand choice functions such that strategy-proof and stable mechanism fail to exist. In our setting, matching without contracts, observable substitutes and observable size monotonicity corresponds to substitutes and size monotonicity. Hence, our discussions on Hatfield and Kojima (2008) also hold here.

To the best of our knowledge, ours is the first paper to introduce the ideas of partitionability, within group substitutes, and downstream independence, and to show that these conditions guarantee a stable assignment. Our paper was first circulated in 2020 under the title "Partitionable Choice Functions." Two papers show similar results: Huang (2021) introduces the idea of unidirectional substitutes and complements. Bando and Kawasaki (2021), in a dynamic model, introduce period-wise substitutability and future invariance. Both papers show the existence of a stable assignment when these conditions are satisfied.

We contrast our concepts to the model of supply chain networks in Ostrovsky (2008). First, we consider a two-sided matching model while Ostrovsky (2008) considers a model of matching with contracts. In the latter it is known that substitutes is not a necessary condition for the existence of a stable assignment (as pointed out by Hatfield and Kojima (2008)). Next, our concept of partitionability identifies a common characteristic in hospitals' preferences over the other side. In their paper this is not possible because of the non-"two-sidedness" of networks: firms have generally different sets of trading partners. Finally, their network structure is given exogenously, while partitions in our environment can arise endogenously.

In terms of mechanisms, Correa et al. (2019) and Dur et al. (2022) also define sequential variants of DA based on exogenously given grades in the school choice problem. The parameter π associated with our SDA is instead endogenously determined by the profile of choice functions (and there can be several). We derive general conditions on profiles of choice functions in which stability is possible. In their environments, choice functions are not primitives and they focus on particular multi-unit demand side "preferences" based on priorities, sibling guarantees, etc.

Our paper is also related to those that study matching problems where both sides' preferences are no longer the only primitives of the model, and additional considerations arise from real-world constraints. We mention only several here. Pathak et al. (2020) present a theory of rationing with reservation amounts for different categories of the population; they discuss their results in the context of vaccine allocation during the Covid-19 crisis.

Kamada and Kojima (2015, 2017) examine the case of distributional constraints, e.g., in the Japan Residency Matching Program, each region may have a maximum number of doctors to ensure that rural areas have enough. Delacrétaz et al. (2019) consider the problem of refugee resettlement, where the standard model fails to recognize characteristics, such as family size, family make-up, languages spoken, etc., important for various quota considerations. Combe et al. (2018) propose a new centralized teacher assignment process in France, where the initial assignment of teachers must be taken into account. In each scenario, known mechanisms like the DA may fail to result in desirable outcomes, or other criteria aside from stability play key roles.

3 Model

Let $M = \{m_1, \ldots, m_k\}$ be a finite set of medical staff. Let $H = \{h_1, \ldots, h_\ell\}$ be a finite set of hospitals. Each hospital h is equipped with a choice function denoted by $C_h : 2^M \to 2^M$ where $C_h(\bar{M}) \subseteq \bar{M}$ is the set of chosen staff, possibly $\emptyset$, by hospital h from menu $\bar{M} \subseteq M$. Let $C = (C_h)_{h \in H}$ be the profile of hospitals' respective choice functions. Each staff $m \in M$ is equipped with a strict preference relation P_m over the set of hospitals and being unassigned, the latter denoted by $\emptyset$; we denote weak preference by R_m. Let $P = (P_m)_{m \in M}$. We define tuple (M, H, P, C) as a **two-sided matching problem**. We fix the sets M and H and represent a problem with (P, C).

A **matching** $\mu : M \to H \cup \{\emptyset\}$ is a function such that each staff is matched with one hospital in H or $\emptyset$.[9] With slight abuse of notation we use $\mu(h)$ instead of $\mu^{-1}(h)$. A **mechanism** ϕ is a procedure which selects a matching for any problem (P, C). Let $\phi[P, C]$, $\phi[P, C](m)$, and $\phi[P, C](h)$ be the matching selected by mechanism ϕ, hospital matched to staff m, and staff matched to hospital h under problem (P, C), respectively.

Stability is a central desideratum in two-sided matching markets, shown to be important for the survival and long-term success of an assignment system (Gale and Shapley, 1962; Roth,

[9]Notice that we do not restrict the number of staff, either doctor or nurse, assigned to a hospital. Instead, each hospital determines its capacity through the choice function.

1982; Roth and Sotomayor, 1992). Formally, a stable matching is defined as follows:

Definition 1. *A matching μ is **stable** if*

- ***(Individual Rationality)*** *$\mu(m)\, P_m\, \emptyset$ and $C_h(\mu(h)) = \mu(h)$ for each $m \in M$ and $h \in H$;*
- ***(No Blocking)*** *there is no hospital h and subset of staff $\bar{M}$ such that $\bar{M} \neq \mu(h)$, $\bar{M} = C_h(\mu(h) \cup \bar{M})$ and $h\, R_m\, \mu(m)$ for all $m \in \bar{M}$.*

We say a matching μ is **within-group stable** if it is individually rational and not blocked by any hospital and set of staff that are all the same type.[10] Notice that, stability implies within-group stability but the reverse does not always hold. A mechanism ϕ is stable (within-group stable) if it selects a stable (within-group stable) matching in any problem. A mechanism ϕ is **strategy-proof for medical staff** if for any problem (P, C) no medical staff benefit from misreporting her preferences, i.e., $\phi[P, C](m)\ R_m\ \phi[P'_m, P_{-m}, C](m)$ for any misreport P'_m, and preference profile P (where $P_{-m} = (P_{\hat{m}})_{\hat{m} \in M \setminus \{m\}}$).

It is well known that existence of a stable matching is guaranteed under certain conditions in many-to-one matching problems. The following is the standard sufficiency condition considered in the literature (Kelso and Crawford, 1982; Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008; Roth and Sotomayor, 1992).[11]

Definition 2. *Hospital h's choice function C_h is **substitutable** if for each set of staff $\bar{M} \subset M$, and each pair $m, m' \in M \setminus \bar{M}$,*

$$m \notin C_h(\bar{M} \cup \{m\}) \Rightarrow m \notin C_h(\bar{M} \cup \{m, m'\}).$$

That is, under a substitutable choice function a staff m rejected in a given subset of staff $\bar{M} \cup \{m\}$ cannot be accepted when a superset of $\bar{M} \cup \{m\}$ is considered. A profile of choice

[10] That is, it is not blocked by any hospital $h \in H$ and subset of staff $\bar{M} \subset M$ such that each staff in $\bar{M}$ belongs to a group.

[11] Both Hatfield and Milgrom (2005) and Hatfield and Kojima (2008) consider the matching with contracts framework. The two-sided matching problem is the special case where there is a unique contract term for each medical staff and hospital pair.

functions C is substitutable if for each $h \in H$, C_h is substitutable.[12]

The following is a consistency property for choice functions introduced by Aygün and Sönmez (2013).[13]

Definition 3. *Hospital h's choice function C_h satisfies* ***independence of rejected alternatives*** *if for each $\bar{M} \subseteq M$, and each $m \in M$, if $m \notin C_h(\bar{M} \cup \{m\})$, then $C_h(\bar{M} \cup \{m\}) = C_h(\bar{M} \setminus \{m\})$.*

That is, removal of the rejected staff does not change the chosen set of the staff. We assume each choice function considered in this paper satisfies independence of rejected alternatives.

Unfortunately, as mentioned in the introduction in our environment hospitals' choice functions may fail to be substitutable. We illustrate more formally in the following example.

Example 1. *Let $H = \{h\}$, $M = \{d, n\}$ where d is a doctor and n is a nurse. Suppose h only accepts the doctor and nurse if they are present as a pair. Then, $C_h(\{n\}) = C_h(\{d\}) = \emptyset$ but $C_h(\{n, d\}) = \{n, d\}$. Hence, C_h is not substitutable.*[14]

Given the negative result in Example 1, we seek a weaker condition that guarantees the existence of a stable matching for any problem.

3.1 A Weaker Condition than Substitutes: Partitionability

In this section, we introduce a condition on choice functions that is weaker than substituability and guarantees existence of a stable matching for any problem. An ordered partition $\pi = \{\pi_1, \pi_2, \dots, \pi_\ell\}$ of staff is a collection of labelled subsets of staff such that $\pi_k \cap \pi_{k'} = \emptyset$ for any $k \neq k'$, and $\cup_{k \leq \ell} \pi_k = M$. For any partition of staff, we refer to a cell of the partition as a group. For instance, in the our medical labor market context the set of staff can be partitioned into nurses and doctors.

[12] Substitutability, as well as weaker conditions, i.e., bilateral and unilateral substitutes (Hatfield and Kojima, 2010), have been shown to be sufficient conditions in the matching with contracts model. Since in our setting each hospital-staff pair have only one possible contract, such conditions are all the same.

[13] Alva (2018) shows that independence of rejected alternatives is equivalent to rationalizability.

[14] Notice that, hospital h's choice function fails to be substitutable even if it accepts a doctor alone.

Definition 4. *Let $\pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$ be an ordered partition of the staff M. A choice function C_h is **π-partitionable** if*

i. (Within-Group Substitutes for π) For each subset of staff $\bar{M} \subset M$, each group $\pi_g \in \pi$, each pair $m, m' \in (M \setminus \bar{M}) \cap \pi_g$,

$$m \notin C_h(\bar{M} \cup \{m\}) \Rightarrow m \notin C_h(\bar{M} \cup \{m, m'\}).$$

ii. (Downstream Independence for π) For each $g < \ell$, let $U_g = \pi_1 \cup \cdots \cup \pi_g$ denote the staff "upstream" of π_g and $O_g = \pi_{g+1} \cup \cdots \cup \pi_\ell$ denote the staff downstream of π_g. For each set of staff $\bar{M} \subset M$, $g < \ell$, and each "downstream" staff $m' \in O_g$,

$$C_h(\bar{M}) \cap U_g = C_h(\bar{M} \cup \{m'\}) \cap U_g.$$

The first condition requires no complementarity between the members of the same group. The second condition says that the presence of downstream staff does not impact which upstream staff are chosen.[15] We also say that a profile of choice functions C is **within-group substitutes for π** if each choice function in C satisfies the first condition, and similarly with **downstream independence for π** for the second condition. For brevity, we drop mention of π when the partition at hand is clear.

Definition 5. *A profile of choice functions C is **partitionable** if there exists an ordered partition of the staff π such that for each $h \in H$, C_h is π-partitionable.*

We make two remarks on this definition.

Remark 1. *Agents from the unit demand side that are in different groups can be either complements (up to some degree) or substitutes. For example, suppose a hospital first chooses doctors then nurses. The hospital might prefer to select nurses with the same specialties as*

[15] This is reminiscent of the *history independence* assumption in Kotowski (2019) in the context of a dynamic matching when agents are repeatedly assigned each period. Their property requires that matchings in previous periods do not affect preferences today. Our *downstream independence* is the opposite—matchings in previous/upstream groups *can* affect selections in the current group, but later/downstream groups cannot affect previous selections.

the doctors they have chosen or instead might prefer to "fill in gaps" and choose nurses that are not in the same field. In either case, the doctors and nurses are not substitutes. Alternatively, a hospital might have "separable" preferences over groups: the highest priority doctors are chosen, then the highest priority nurses are chosen, with the priorities being independent. In this case, agents across the two groups are substitutes. Finally, note that perfect complements across groups is still not permissible: The choice function in Example 1, where either the entire "team" of the doctor-nurse pair is accepted together or not at all, is not partitionable.

Remark 2. *Partitionability need not be based an exogenous characteristic. In many applications, there is a structure on applicants that generates a natural partition, e.g., a doctor's specialty or different types of medical staff. However, partitionability is well-defined even when there are no exogenous characteristics. Given any arbitrary profile of choice functions, there may be a partition that "rationalizes" the profile of choice functions as partitionable.*

We first show the relation between partitionability and substitutability.

Proposition 1. *If a profile of choice functions is substitutable, then it is partitionable. In contrast, a partitionable profile of choice functions is not necessarily substitutable.*

3.2 Partitionability and Stability

Proposition 1 implies that partitionability is a weaker condition than substitutability. Yet, as shown in the next theorem, it is sufficient to guarantee existence of stable matching.

Theorem 1. *If a profile of choice functions C is partitionable, then for any problem (P, C), a stable matching exists.*

We prove Theorem 1 via construction. We first define a family of mechanisms used in our proof. This family of mechanisms is parameterized by ordered partitions $\pi = \{\pi_1, \pi_2, \ldots, \pi_\ell\}$—each called the **Sequential Deferred Acceptance w.r.t. $\boldsymbol{\pi}$** (SDA^π).

Let (P, C) be a problem, and π be an ordered partition of the medical staff such that C is π-partitionable. Define the outcome of SDA^{π} as the product of the following algorithm:

Step 1: Deferred Acceptance for $\boldsymbol{\pi_1}$

Step 1.1: Each medical staff $m \in \pi_1$ applies to her top choice under P_m. Let $A_1(h)$ be the set of staff who have applied to h in this step. Each hospital h tentatively keeps the staff in $C_h(A_1(h))$ and rejects the staff in $A_1(h) \setminus C_h(A_1(h))$.

In general for each $k > 1$:

Step 1.*k*: Each medical staff $m \in \pi_1$ applies to her top choice under P_m which has not rejected her yet. Let $A_k(h)$ be the set of staff who have applied to h in this step. Each hospital h tentatively keeps the staff in $C_h(A_k(h))$ and rejects the staff in $A_k(h) \setminus C_h(A_k(h))$.

Step 1 terminates when no staff in π_1 is rejected. For each $h \in H$, let $\mu_1(h)$ be the set of staff tentatively kept by h at the last step.

In general, for each $\ell \geq g > 1$:

Step *g*: Deferred Acceptance for $\boldsymbol{\pi_g}$

Step *g*.1: Each medical staff $m \in \pi_g$ applies to her top choice under P_m. Let $A_1(h)$ be the set of doctors who have applied to h in this step. Each hospital h tentatively keeps the staff in $C_h(\mu_{g-1}(h) \cup A_1(h))$ and rejects the staff in $A_1(h) \setminus C_h(\mu_{g-1}(h) \cup A_1(h))$.[16]

In general for each $k > 1$:

Step *g*.*k*: Each medical staff $m \in \pi_g$ applies to her top choice under P_m which has not rejected her yet. Let $A_k(h)$ be the set of staff who have applied to h in this step. Each hospital h tentatively keeps the staff in $C_h(\mu_{g-1}(h) \cup A_k(h))$ and rejects the staff in $A_k(h) \setminus C_h(\mu_{g-1}(h) \cup A_k(h))$.

[16] Note that by downstream independence, $\mu_{g-1}(h)$ is a subset of the chosen staff from $\mu_{g-1}(h) \cup A_1(h)$.

Step g terminates when no staff in π_g is rejected. For each $h \in H$, let $\mu_g(h)$ be the set of staff tentatively kept by h at the last step.

The Deferred Acceptance (DA) mechanism is the special case when $\pi = \{M\}$. In the Appendix, we establish that the outcome of this algorithm is a stable matching, thereby proving Theorem 1. For the rest of the paper, all omitted proofs are in the Appendix.

In Hatfield and Kojima (2010), they consider one hospital with a choice function that violates substitutes. They show that if there is at least one more hospital and this hospital's choice function satisfies substitutes, then a stable matching need not exist. We will perform the parallel exercise with partitionability and show that departure from partitionability may result in the nonexistence of a stable matching.

Proposition 2. *Let $|H| \geq 2$ and $|M| \geq 2$. If there is $h \in H$ such that C_h is not partitionable, then we can find a preference profile P for staff and a partitionable choice function for some hospital h' (with unit demand for each group) such that, regardless of the choices of the other hospitals, at (P, C) there is no stable matching.*

We conclude this section by providing another feature of SDA^π. In particular, we provide a condition such that when hospitals' choice functions satisfy it (in addition to partitionability), SDA^π is immune to preference manipulation by medical staff.

Definition 6. *Hospital h's choice function C_h satisfies the **law of aggregate demand** if $M' \subset M''$ implies $|C_h(M')| \leq |C_h(M'')|$.*

That is, when a hospital takes additional staff into consideration, the number of chosen staff weakly increases. A profile of choice functions C satisfies the **law of aggregate demand** if for each $h \in H$, C_h satisfies the law of aggregate demand. Hatfield and Milgrom (2005) show that when the multi-unit demand side's choice functions satisfy the law of aggregate demand and substitutability conditions, then the DA mechanism cannot be manipulated by agents on the unit demand side. In Proposition 3 we show that SDA^π cannot be manipulated when hospital choice functions are partitionable and satisfy the law of aggregate demand.

Proposition 3. *If a profile of choice functions C is partitionable and satisfies the law of aggregate demand, then SDA^{π} is strategy-proof for medical staff.*

The statement also holds for a relaxation of the law of aggregate demand where we further require that all staff in $M'' \setminus M'$ are in the same group. By partitionability, at Step k the assignment is independent of agents in Step k' for $k' > k$, and the result follows.

4 Extension: Pure Complements

Consider the problem of matching hospitals and medical staff composed of doctors and nurses. Let D be the set of doctors, N be the set of nurses, and $M = D \cup N$. A hospital wishes to hire a doctor only if it can support her with $\alpha \in \mathbb{N}$ nurses, and similarly, to hire a nurse only to support a doctor.[17]

Formally, let a **ratio choice function** $C_h(\cdot)$ satisfy *within-group substitutability* and be such that for each $M' = D' \cup N' \in 2^M$,

- (α Nurses to Doctor Ratio) $\alpha|C_h(M') \cap D| = |C_h(M') \cap N|$
- (Doctor Selection Partial Independence) for each $\bar{N} \subseteq N$ with $|\bar{N}| = |N'|$, $C_h(D' \cup N') \cap D = C_h(D' \cup \bar{N}) \cap D$.

A ratio choice function is not substitutable. If $\alpha = 1$, then we have the scenario from Example 1: it is possible that $C_h(\{d\}) = C_h(\{n\}) = \emptyset$ while $C_h(\{d, n\}) = \{d, n\}$. Further note that it is not partitionable as it fails the downstream independence condition (and hence we refer to it as "partial" independence). Recall that the latter requires the availability of downstream staff to not affect the selection of upstream staff. Whether we consider doctors or nurses first in the ordered partition, adding the second agent causes the first agent to be chosen when they were rejected before. Thus, even with its built-in structure, a ratio choice function exhibits "pure complements" choice behavior and fails partitionability.

[17]Our result in this section still holds if the number of nurses needed to support a doctor differs across doctors.

Despite this, we provide conditions on the domain of problems where our results can still be useful. Say that all nurses are **acceptable** to hospital h if for each $M' \subseteq M$, and each $\bar{N} \subseteq N$, if $|\bar{N}| = \alpha|C_h(M') \cap D|$, then $\bar{N} \subseteq C_h((C_h(M') \cap D) \cup \bar{N})$. Any set of chosen nurses can be swapped with another set of nurses of the same cardinality. Consider the case where nurses are not in shortage and find all hospitals acceptable. In this scenario, if we run the SDA with doctors selected first, then these conditions guarantee that eventually each hospital h will have a sufficient number of nurses applying to h. A hospital will then not regret hiring any doctor, as it can find α nurses for each.

SDA with Placeholders:

Step 1. (Doctor Assignment) Use the DA algorithm, and in each substep k:

i. Each doctor applies to her most preferred hospital that has not rejected her yet.

ii. Let D_h^k be the set of doctors applying to hospital h in substep k. Each hospital h tentatively keeps doctors in $C_h(D_h^k \cup N)$ and rejects the rest of the doctors in D_h^k.

The DA algorithm terminates when no doctor is rejected. When it terminates, we assign *only* the chosen doctors to the hospitals. Let D_h be the set of doctors assigned to hospital h at the end of Step 1.

Step 2. (Nurse Assignment) Use the DA algorithm, and in each substep k:

i. Each nurse applies to her most preferred hospital that has not rejected her yet.

ii. Let N_h^k be the set of nurses applying to hospital h in substep k. Each hospital h with $|N_h^k| > \alpha|D_h|$ tentatively keeps nurses in $C_h(D_h \cup N_h^k)$ and rejects the rest of the nurses in N_h^k. Each hospital h with $|N_h^k| \leq \alpha|D_h|$ tentatively keeps all nurses in N_h^k.

The DA algorithm terminates when no nurse is rejected. The nurses applying to hospital h in the last substep and the doctors in D_h are assigned to hospital h.

In Step 1, we run DA as if each hospital has nurses available; the hospitals use these "placeholder" assignments to make their doctor selections. After Step 2, these placeholder assignments are discarded, and replaced with the actual assignments.

It is straightforward to see that SDA with Placeholders works as desired when there is no nurse scarcity. As with the SDA, it selects a stable matching. Furthermore, any ratio choice function satisfies the law the aggregate demand, resulting in the mechanism's non-manipulability by the medical staff. Arguments parallel exactly the proofs of Theorem 1 and Proposition 3. We summarize this finding next.[18]

Theorem 2. *Let (P, C) be a problem such that*

i. C is a profile of ratio choice functions,

ii. $|N| \geq \alpha|D|$, and

iii. each nurse finds each hospital acceptable and each hospital finds all nurses acceptable.

Then, for any such problem, SDA with Placeholders selects a stable matching, and is strategy-proof for the medical staff.

Theorem 2 shows that we can achieve stability when doctors are scarce relative to nurses. We illustrate how the statement fails if nurses are scarce.[19]

Example 2. *Let $H = \{h_1, h_2\}$, $N = \{n_1, n_2, n_3\}$, $D = \{d_1, d_2\}$, and $\alpha = 3$. In this problem, we have $|N| = 3 < 6 = \alpha|D|$. Each hospital has a ratio choice function and would hire d_1 with α nurses over d_2 with the same. Preferences of doctors are as follows: $h_1 \; P_{d_1} \; h_2$ and $h_2 \; P_{d_2} \; h_1$. Preferences of each nurse n is $h_1 \; P_n \; h_2$. Then, SDA with Placeholders will assign: n_1, n_2, n_3 and d_1 to h_1, and d_2 to h_2. This assignment is not stable because it is not individually rational for h_2.*

[18]For completeness, we provide the proof of Theorem 2 in the Appendix.

[19]A similar problem occurs if we attempt to reverse the order of SDA with Placeholders and assign nurses first by permuting the roles of the two groups.

5 Conclusion

Substitutability is sufficient to guarantee the existence of a stable assignment. However, partitionability is also sufficient and a weaker condition than substitutability. Therefore, we conclude that substitutability is not necessary for the existence of a stable assignment.

If one choice function violates substitutes, we can always find a choice function that satisfies substitutes but where a stable assignment need not exist. We showed the analagous statement with partitionability. Again, since partitionability is a weaker condition than substitutability, we conclude that even under this interpretation of necessity, substitutability is not necessary for the existence of an equilibrium.

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Appendix: Proofs

Proof of Proposition 1. Let π be the trivial partition consisting of all staff, i.e. $\pi = \{M\}$. Then, a substitutable choice function C_h is π-partitionable. Hence, $C = \{C_{h_1}, \ldots, C_{h_\ell}\}$ is partitionable.

Next, we prove the second part of the statement by means of example. Let $H = \{h\}$, $\pi = \{\pi_1, \pi_2\}$, $\pi_1 = \{d\}$ and $\pi_2 = \{n\}$. That is, $M = \{d, n\}$. Here, d is a doctor and n is a nurse. We define C_h as follows:

a. Select n only when d is also applying.

b. Select d whenever she is applying.

Such a choice function can be interpreted as follows: A doctor can do a job without a nurse. However, in order a nurse to help a patient, we need a doctor. A doctor and a nurse together work more efficiently.

First notice that, C_h is not substitutable: $n \notin C_h(\{n\}) = \emptyset$ but $n \in C_h(\{d, n\}) = \{d, n\}$.

However, C_h is π-partitionable: d is never rejected when she applies and n in π_2 is accepted only when d in π_1 applies. That is, both within-group substitutes and downstream independence are satisfied. □

Proof of Theorem 1. We show that if a profile of choice functions is π-partitionable, then the SDA^π produces a stable matching for any staff preference profile.

Let (P, C) be a problem, and $\pi = (\pi_1, \ldots, \pi_\ell)$ be an ordered partition of the medical staff such that C is π-partitionable. Let the SDA^π prescribe μ for this problem. Since the choice function for each hospital h is π-partitionable, $\mu_g(h) \subseteq \mu_{g+1}(h)$ for any $g \in \{1, 2, \ldots, \ell - 1\}$. Hence, $\mu(h) = \mu_\ell(h)$ for each $h \in H$.

We now show that μ is stable for this problem. We first consider individual rationality. Since staff only apply for the acceptable hospitals, $\mu(m)\ R_m\ \emptyset$ for all $m \in M$. Moreover, independence of rejected alternatives implies that for each $h \in H$, $C_h(\mu_{\ell-1}(h) \cup A_{\bar{k}}(h)) =$

$C_h(\mu_\ell(h)) = \mu_\ell(h) = \mu(h)$ where $\bar{k}$ is the termination step of the DA for π_ℓ. Hence, μ is individually rational.

Next, we show μ cannot be blocked. Suppose by contradiction that there is a hospital h and a set of staff $\bar{M} \subseteq M$ that block μ. So $\bar{M} \setminus \mu(h) \neq \emptyset$ and $\bar{M} \subseteq C_h(\mu(h) \cup \bar{M})$. We prove this cannot be true by using induction. Let $\bar{M} \cap \pi_g = \bar{M}^g$ and $\mu(h) \cap \pi_g = \mu^g(h)$ for each $g \in \{1, 2, \ldots, \ell\}$ and $h \in H$. We start with $g = 1$. If there exists $m \in \bar{M}^1 \setminus \mu^1(h)$, then by independence of rejected alternatives and partitionability, $m \notin C_h(\mu(h) \cup \bar{M})$—contradicting the assumption that $\bar{M}$ blocks μ. Hence, $\bar{M}^1 \subseteq \mu^1(h) \subseteq \mu(h)$. By independence of rejected alternatives and downstream independence, $\bar{M}^1 \subseteq C_h(\mu(h) \cup \bar{M})$. Suppose $\bar{M}^{g'} \subseteq \mu(h)$ for all $g' < \bar{g} \leq \ell$. That is, $\bar{M}^{g'} \subseteq C_h(\mu(h) \cup \bar{M})$ for all $g' < \bar{g} \leq \ell$. If there exists $m \in \bar{M}^{\bar{g}} \setminus \mu^{\bar{g}}(h)$, then by independence of rejected alternatives and partitionability, $m \notin C_h(\mu(h) \cup \bar{M})$. Hence, $\bar{M}^{\bar{g}} \subseteq \mu^{\bar{g}}(h)$. Therefore, we have $\bar{M} \subseteq \mu(h)$ and therefore $C_h(\mu(h) \cup \bar{M}) = C_h(\mu(h)) = \mu(h)$. □

Proof of Proposition 2. Fix any partition π. Since C_h is not partitionable, C_h is not π-partitionable. The definition of partitionability includes two independent properties.

Within-Group Substitutes: Suppose C_h violates within-group substitutes. That is, there is a group π_g, and staff $\{m, j\} \subseteq \pi_g$ and $\bar{M} \subset M$ such that $m \notin C_h(\bar{M} \cup \{m\})$ but $m \in C_h(\bar{M} \cup \{m, j\})$. We will show that $j \in C_h(\bar{M} \cup \{m, j\})$. Otherwise, $C_h(\bar{M} \cup \{m, j\}) = C_h(\bar{M} \cup \{m\})$ (by the independence of rejected alternatives), and since $m \notin C_h(\bar{M} \cup \{m\})$, $m \notin C_h(\bar{M} \cup \{m, j\})$, a contradiction. Therefore, $\{m, j\} \subseteq C_h(\bar{M} \cup \{m, j\})$.

We will construct a partitionable choice function for some hospital h'. Specifically, for any set $M' \subseteq M \setminus \{m, j\}$, let $C_{h'}(M') = \emptyset$, $C_{h'}(M' \cup \{j\}) = \{j\}$ and $C_{h'}(M' \cup \{m\}) = C_{h'}(M' \cup \{m, j\}) = \{m\}$.

Let $M^* = C_h(\bar{M} \cup \{m, j\})$ recalling that $\{m, j\} \subseteq M^*$. Let $\tilde{M} = C_h(\bar{M} \cup \{m\})$. By irrelevance of rejected alternatives, $\tilde{M} = C_h(\bar{M} \cup \{m\}) = C_h(\tilde{M} \cup \{m\})$. We now define staff preferences as follows:

$$P_k := \begin{cases} h, h', \emptyset & k = m \\ h', h, \emptyset & k = j \\ h, \emptyset & k \in \bar{M} \setminus \{m, j\} \\ \emptyset & k \notin \bar{M} \cup \{m, j\} \end{cases}$$

Suppose by contradiction that there is a stable matching μ at this problem. Then, $\mu(k) = \emptyset$ for all $k \notin \bar{M} \cup \{m, j\}$ and $\mu(h') \subseteq \{m, j\}$. Otherwise, μ is blocked by an individual staff or hospital. If $\mu(h') = \emptyset$, then h' and j block μ. Hence, $|\mu(h')| = 1$. If $\mu(h') = m$, then staff in M^* and h block μ. If $\mu(h') = j$, then $\mu(m) = h$. Otherwise, m and h' block μ. Since $m \notin C_h(\tilde{M} \cup \{m\}) = \tilde{M}$, $\tilde{M} \setminus \mu(h) \neq \emptyset$. Hence, staff in $\tilde{M}$ and h block μ. In any case, μ is blocked.

Downstream Independence: Suppose C_h violates downstream independence. That is, for some group π_g, $\bar{M} \subset M$, and $j \in O_g \setminus \bar{M}$, we have $C_h(\bar{M}) \cap U_g \neq C_h(\bar{M} \cup \{j\}) \cap U_g$.

Case 1: A downstream doctor causes an upstream staff to be rejected. Formally, suppose $m \in C_h(\bar{M}) \cap U_g$ but $m \notin C_h(\bar{M} \cup \{j\}) \cap U_g$. Similar to the argument in the within-group substitutes section, irrelevance of rejected alternatives implies that $j \in C_h(\bar{M} \cup \{j\})$. For hospital h', define $C_{h'}$ as the following: for any $I' \subseteq M \setminus \{m, j\}$, $C_{h'}(I') = C_{h'}(I' \cup \{j\}) = \emptyset$, $C_{h'}(I' \cup \{m, j\}) = \{m, j\}$, and $C_{h'}(I' \cup \{m\}) = \{m\}$. Note that even though $C_{h'}$ violates substitutes, it is π-partitionable if m is in an earlier group than j. Let $M^* = C_h(\bar{M} \cup \{j\})$ recalling that $j \in M^*$. By irrelevance of rejected alternatives, $C_h(\bar{M} \cup \{j\}) = C_h(M^*)$. We now define staff preferences as follows:

$$P_k := \begin{cases} h, h', \emptyset & k = m \\ h', h, \emptyset & k = j \\ h, \emptyset & k \in \bar{M} \setminus \{m, j\} \\ \emptyset & k \notin \bar{M} \cup \{m, j\} \end{cases}$$

Suppose by contradiction that there is a stable matching μ. Then, $\mu(k) = \emptyset$ for all $k \notin \big(\bar{M} \cup \{m, j\}\big)$. If $\mu(m) = h'$, then $\mu(j) = h'$ or else $\{m, j\}$ and h' block μ. But if $\mu(m) = \mu(j) = h'$, then $\mu(h) \neq C_h(\bar{M})$ and staff in $C_h(\bar{M})$ and h block μ. If $\mu(m) = \emptyset$, then m and h' block μ. Therefore, $\mu(m) = h$ and $\mu(h) \neq M^*$. In this case, $\mu(j) \neq h'$ as $j \notin C_{h'}(\{j\})$. Moreover, by irrelevance of rejected alternatives of C_h and individual rationality of μ, $M^* \nsubseteq \mu(h)$ as $m \notin M^* = C_h(\tilde{M} \cup \{j\})$ for any $\tilde{M} \subseteq M$ such that $M^* \subseteq \tilde{M} \subseteq \bar{M}$. Hence, the staff in M^* and h block μ. In any case, μ is blocked.

Case 2: A downstream doctor causes an upstream staff to be accepted. Formally, suppose $m \notin C_h(\bar{M} \cup \{m\}) \cap U_g$ but $m \in C_h(\bar{M} \cup \{m, j\}) \cap U_g$. By irrelevance of rejected alternatives, $\{m, j\} \subseteq C_h(\bar{M} \cup \{m, j\})$. Notice that C_h violates the substitutes condition.

We will construct a partitionable choice function for some hospital h'. Specifically, for any set $M' \subseteq M \setminus \{m, j\}$, let $C_{h'}(M') = \emptyset$, $C_{h'}(M' \cup \{j\}) = \{j\}$ and $C_{h'}(M' \cup \{m\}) = C_{h'}(M' \cup \{m, j\}) = \{m\}$.

Let $M^* = C_h(\bar{M} \cup \{m, j\})$ recalling that $\{m, j\} \subseteq M^*$. Let $\tilde{M} = C_h(\bar{M} \cup \{m\})$. By irrelevance of rejected alternatives, $\tilde{M} = C_h(\bar{M} \cup \{m\}) = C_h(\tilde{M} \cup \{m\})$. We now define doctor preferences as follows:

$$P_k := \begin{cases} h, h', \emptyset & k = m \\ h', h, \emptyset & k = j \\ h, \emptyset & k \in \bar{M} \setminus \{m, j\} \\ \emptyset & k \notin \bar{M} \cup \{m, j\} \end{cases}$$

Suppose by contradiction that there is a stable matching μ. Then, $\mu(k) = \emptyset$ for all $k \notin \big(\bar{M} \cup \{m, j\}\big)$ and $\mu(h') \subseteq \{m, j\}$. Otherwise, μ is blocked by an individual staff or hospital. If $\mu(h') = \emptyset$, then h' and j block μ. Hence, by individual rationality, $|\mu(h')| = 1$. If $\mu(h') = m$, then $\mu(h) \neq M^*$ and doctors in M^* and h block μ. If $\mu(h') = j$, then $\mu(m) = h$. Otherwise, m and h' block μ. Since $m \notin C_h(\tilde{M} \cup \{m\}) = \tilde{M}$, $\mu(h) \neq \tilde{M}$. By individual rationality of μ, $\mu(h) \subseteq \bar{M} \cup \{m\}$, and so $\mu(h) \cup \tilde{M} \subseteq \bar{M} \cup \{m\}$. By independence of irrelevant alternatives,

$C_h(\bar{M} \cup \{m\}) = C_h(\mu(h) \cup \tilde{M}) = \tilde{M}$. Hence, staff in $\tilde{M}$ and h block μ. In any case, μ is blocked. $\square$

Proof of Proposition 3. Consider an arbitrary problem (P, C). The SDA^{π} mechanism assigns each partition π_k in Step k to the hospitals. Downstream independence implies that staff in π_k cannot affect the assignment determined in earlier steps. Hence, by Hatfield and Milgrom (2005), since hospitals' choice function satisfies within-group substitutes and the law of aggregate demand, in any Step k staff in π_k cannot benefit from misreporting their preferences over hospitals and being unassigned option. As a result, SDA^{π} is strategy-proof for medical staff. $\square$

Proof of Theorem 2. First, when Conditions i and iii are satisfied the SDA with Placeholders selects a well-defined matching, i.e., no medical staff is assigned to more than one hospital. In particular, in Step 1 no doctor is applying to multiple hospitals at the terminal substep; in Step 2, no nurse is applying to multiple hospitals at the terminal substep.

In Step 2, Condition ii ensures that each hospital h which has been assigned $|D_h|$ doctors will end up with $\alpha|D_h|$ nurses. In particular, in any substep of Step 2, hospital h rejects a nurse only if more than $\alpha|D_h|$ nurses has applied to it. Conditions i and ii and the proof of Theorem 1 thus guarantee that the outcome obtained at the end of Step 2 is stable.

For each hospital h, choice function C_h satisfies the law of aggregate demand. Hence, Conditions $i - iii$ and the proof of Proposition 3 implies that no medical staff can manipulate SDA with Placeholders. $\square$